Infinite Games

Summer course at the University of Sofia, Bulgaria

Yurii Khomskii

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0 Introduction

0.1 What this course is about

In this course, we will give an introduction to the mathematical theory of *two-person perfect-information games of infinite length.* The systematic study of mind games such as chess, checkers and Go was begun in 1913 by Ernst Zermelo and carried on by various other mathematicians. Of course, real-life games always have a finite length, i.e., the games are bound to end after some finite number of turns—otherwise it would not be possible to play them. However, once we have a mathematical formalism that works for finite games, we can easily extend it to include *infinite games*, that is, games similar to chess, Go etc., but where the number of moves is infinite: the game goes on "forever", so to say.

Why would anyone want to study such a thing? Even if it is mathematically feasible, it might look like a theory of infinite games would have no applications whatsoever. Surprisingly, this is far from being true, and indeed the theory of infinite games has remarkably many applications in various areas of logic and pure mathematics, including, in particular, set theory, basic topology and the study of the real number continuum.

Of course, the shift from finite to infinite games also involves a shift in perspective: we are no longer using a mathematical theory to study real-life games (interesting in themselves), but rather *creating* or *constructing* new games, the study of which may help us to understand fundamental mathematical objects. The games change from being the subjects of research to being tools in the study of other subjects.

We hope that this course will convey an appreciation for the importance and usefulness of infinite games.

0.2 Notation and prerequisites

We will assume familiarity with basic mathematical notation and knowledge of mathematical structures, in particular *infinite sets*, the set of all functions from one set to another, etc. We assume that the readers know terms such as "surjection", "injection", "transitive relation" and so on. We also assume familiarity with the notions of *cardinality* of a set. We write $|A|$ to denote the cardinality of A.

Our main mathematical starting-point is *the set of all natural numbers* $\{0, 1, 2, 3, \dots\}$, which we denote by ω. For a number n, ω^n is the n-Cartesian product of ω, i.e., $\omega \times \cdots \times \omega$ repeated n times. We will denote elements of ω^n by $\langle x_0, x_1, \dots, x_{n-1} \rangle$, where x_i is a natural number, and call these *finite sequences.* For practical purposes, however, we will frequently identify a finite sequence with a function $f : \{0, \dots, n-1\} \longrightarrow \omega$, so that if $f \in \omega^n$, we can write $f(m)$ to denote the m-th element of the sequence f. We will usually use the letters s, t etc. for finite sequences. The *empty sequence* is denoted by $\langle\rangle$.

The set of *all finite sequences* is denoted by

$$\omega^{<\omega} = \bigcup_{n\in\omega} \omega^n$$

and for $s \in \omega^{<\omega}$, $|s|$ is the *length of* s, i.e., $|s| = n$ such that $s \in \omega^n$.

Generalizing this to an infinite Cartesian product, we have the set

$$\omega^\omega := \{f : \omega \longrightarrow \omega\}$$

So ω^ω is the set of all functions from the natural numbers to the natural numbers. We can view such functions as *infinite sequences of natural numbers*, and for that reason we shall sometimes use a notation like $\langle 0, 0, 0, \dots \rangle$, or $\langle x_0, x_1, \dots \rangle$. We will usually use the letters x, y etc. for elements of ω^ω.

If $x \in \omega^\omega$ and $n \in \omega$, then $x \restriction n$ denotes the *initial segment of* x *of length* n, to be precise

$$x \restriction n := \langle x(0), x(1), \dots, x(n-1) \rangle$$

If $s \in \omega^{<\omega}$ and $x \in \omega^\omega$, then we say that s *is an initial segment of* x, denoted by

$$s \lhd x$$

if for all $m < |s|$ we have $s(m) = f(m)$, equivalently if $x \restriction |s| = s$. For two finite sequences $s, t \in \omega^{<\omega}$, the *concatenation* of s and t, denoted by $s^\frown t$, is defined as expected: if $|s| = n$ and $|t| = m$ then

$$s^\frown t := \langle s(0), s(1), \dots, s(n-1), t(0), t(1), \dots, t(m-1) \rangle$$

An equivalent definition is

$$s^\frown t(i) := \begin{cases} s(i) & \text{if } i < n \\ t(i-n) & \text{if } i \geq n \end{cases}$$

for all $i < n + m$. By analogy, for $s \in \omega^{<\omega}$ and $x \in \omega^\omega$ we define the concatenation of s and x by

$$s^\frown x := \langle s(0), s(1), \dots, s(n-1), x(0), x(1), \dots \rangle$$

Any piece of notation not included in the list above will be defined as we go along.

1 Finite games

1.1 Our basic setting

We begin the story in the most natural place: the study of real-life, finite games. We will present the general setting, derive a mathematical formalism and then show how it applies to some concrete examples such as chess. Although this is not the final objective of our study, it is nevertheless important to understand how the paradigm of infinite games is developed out of the one for finite games.

We consider the following setting: there are two players, called *Player I* and *Player II*, who are playing a turn-based mind game (say, with pieces on a board) against each other. By convention, Player I is a male player whereas Player II is female. Player I always starts the game by making some move, after which Player II makes a move, and so they both take turns in playing the game. At some point the game is finished and either Player I has won the game or Player II has. For simplicity we only consider so-called *zero-sum* games, meaning that exactly one of the two players wins the game, and there are no draws. In other words

Player I wins a game if and only if
Player II loses the game, and vice versa.

As this is not the case with many two-player games, we must do a little bit of tweaking: change the rules of each game so that a draw will signify a win for one of the players (for example, a draw in chess is a win for black). This restriction is necessary from a purely technical point of view, and later on we shall see that this restriction is easily overcome and does not affect the kinds of games we can model.

Our other main assumption is *perfect information*, which refers to the fact that both Players I and II have complete access to and knowledge of the way the game has been played so far.

Our framework includes such famous and popular mind games as chess, checkers, Go and tic-tac-toe; and also a variety of less widely known games (reversi, nine men's morris or Go-moku are some other examples). Nevertheless, many games do not fall into our framework. Specifically, we do not consider the following:

1. Games that contain an element of chance, such as throwing a die or dealing cards. For example, backgammon, poker, roulette etc. are not included.
2. Games in which two players take turns *simultaneously* (or so quickly following one another that they have no chance to react to the opponent's move), such as Rock-Paper-Scissors.
3. Games in which certain moves are only known to one player but hidden from the other, such as Stratego.

1.2 Chess

Let us start by a (still informal) discussion of the game of chess. As already mentioned, we alter the rules of chess so that a draw is considered a win for Black. It is clear that, with this condition, chess fulfills all our requirements: there are two players—White and Black in this case—White starts and then the players alternate in taking turns. At each stage, the players have perfect information about the preceding moves. At the end, either White wins, or Black wins, or there is a draw—which again means that Black wins.

Why is chess a finite game? The reason lies in the following simple rule: if a position on the board is repeated three times, with the same player having to go, then the game is automatically called a draw. The number of positions in chess is finite: there are 64 squares, each can be occupied by at most one piece, and there are 32 different chess-pieces, so there are at most 64^{33} positions. Thus the game cannot take longer than $2 \cdot 3 \cdot 64^{33}$.

In fact, we could easily get a much smaller estimate if we took into account how many pieces of each kind there are in chess, that some pieces are identical and do not need distinguishing between each other, that many combinations of pieces on the board are not even legal, and so on. But we are decidedly uninterested in questions of real-life (computational) complexity, and for our purposes any finite number is as good as another.

How can we model or formalize the game of chess? Obviously there are many ways. The most natural one, perhaps, is to use *algebraic chess notation.* Each game of chess can then be written down as a sequence of moves. Below is an example of a short game of chess (*scholar's mate*):

White:	e4		Qh5		Bc4		Qxf7#
Black:		e5		Nc6		Nf6	

An alternative way would be to assign a natural number between 0 and 64^{33} to each unique position of the pieces on the board, and to write the *positions*, rather than the *moves*, in a table analogous to the one above:

White:	x_0		x_1		x_2		x_3	
Black:		y_0		y_1		y_2		$\dots$

In either cases, we require that each step in the game corresponds to a legal move according to the rules of chess, and when the game ends, there is a clear algorithm for determining who the winner is. Using the first formalism, this is incorporated into the notation (a "#" means "check-mate"), whereas in the second one, certain numbers n correspond to a winning or losing position.

One could think of many other ways of encoding a game of chess. Regardless which method we use, each completed game of chess is encoded as a *finite sequence* of natural numbers of length at most 64^{33}. In other words, each game is an elements of ω^n for some $n \leq 64^{33}$. Let LEGAL be the set of those finite sequences that correspond to a sequence of legal moves according to the rules of chess (keeping in mind the particular encoding we have chosen). Now let $\mathsf{WIN} \subseteq \mathsf{LEGAL}$ be the subset of all such sequences which encode a winning game for White. Clearly $\mathsf{LEGAL} - \mathsf{WIN}$ is the set of legal games that correspond to a win by Black. Thus, once the formalism of encoding moves of chess into natural numbers has been agreed upon, the game called "chess" is completely determined by the sets LEGAL and WIN.

1.3 General finite games

After an informal introduction to the mathematization of chess, we now give a precise definition of finite games, as a natural abstraction from the particular case, and at the same time paving the way to the introduction of infinite games in the next chapter.

1.3.1 Definition. (Two-person, perfect-information finite game.) Let N be a natural number (the *length* of the game), and A an arbitrary subset of ω^{2N}. The game $G_N(A)$ is played as follows:

- There are two players, Player I and Player II, which take turns in picking one natural number at each step of the game.
- At each turn i, we denote Player I's choice by x_i and Player II's choice by y_i.
- After N turns have been played the game looks as follows:

$$\begin{array}{r||cccccccc} \text{I:} & x_0 & & x_1 & & \dots & & x_{N-1} & \\ \hline \text{II:} & & y_0 & & y_1 & & \dots & & y_{N-1} \end{array}$$

  The sequence $s := \langle x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1}\rangle$ is called a *play of the game* $G_N(A)$.

- Player I *wins the game* $G_N(A)$ if $s \in A$, and Player II wins if $s \notin A$. The set A is called the *pay-off set* for Player I or the *set of winning conditions* for Player I.

If we look at this definition we immediately notice two aspects in which it differs from the informal discussion of chess in the last section. Firstly, we are only considering sequences of length *exactly* $2N$ as valid plays, rather than sequences of length less then or equal to $2N$. Secondly, instead of restricting the possible plays to some given set (like we defined LEGAL before) and only considering those sequences, we allow *any* sequence of natural numbers of length $2N$ to be considered as a valid play.

As it turns out, this change of formalism does not restrict the class of games we can model. The first problem is easily fixed simply by assigning one particular natural number (say 0) to represent the state in which "the game has been completed". For example, suppose a particular game of chess only took 20 moves, but our model requires the game to be N moves long (for $N = 64^{33}$, say). Then we simply fill in 0's for all the moves after the 20-th until the N-th. It is clear that this allows us to model the same class of games.

To fix the second problem, let us think about the following situation: suppose in a game of chess, a player makes an illegal move, i.e., a move that is not allowed by the rules of chess. One could then do one of two things: tell the player that the move was illegal and should be re-played, or (in a stricter environment) disqualify the player immediately, thus making him or her lose that game. In our mathematical formalism, we chose the second option: so instead of stipulating that only certain moves are allowed, we allow *all* possible moves to be played but make sure that any player who makes an illegal move immediately loses the game. That information can be encoded in the pay-off set $A \subseteq \omega^{2N}$.

The main reason we chose this formalism rather than the one involving LEGAL is purely technical: it is much easier to work with one set A rather than a combination of two sets. This will become especially clear after we have extended finite games infinite games.

Note that we do not put any upper bound on the height of the individual numbers x_i and y_i played by each player. Therefore, Definition 1.3.1 also allows us to model games with an infinite number of possible moves (for example, games on an infinite board), as long as there is a limit to the length of the game.

1.4 Strategies

So far, we have only discussed a convenient mathematical abstraction of finite games, but we have not seen anything of mathematical importance yet. The main concept in the study of games (finite and infinite) is that of a *strategy*. Informally, a strategy for a player is a method of determining the next move based on the preceding sequence of moves. Formally, we introduce the following definition:

1.4.1 Definition. Let $G_N(A)$ be a finite game of length N. A *strategy for Player I* is a function

$$\sigma : \{s \in \bigcup_{n<2N} \omega^n \ : \ |s| \text{ is even } \} \longrightarrow \omega$$

A *strategy for Player II* is a function

$$\tau : \{s \in \bigcup_{n<2N} \omega^n \ : \ |s| \text{ is odd } \} \longrightarrow \omega$$

So a strategy for Player I is a function assigning a natural number to any even sequence of natural numbers, i.e., assigning the next move to any sequence of preceding moves, and the same holds for Player II. Note that it is Player I's turn to move if and only if the sequence of preceding moves is even, and Player II's turn if and only if it is odd.

Given a strategy σ for Player I, we can look at any sequence $t = \langle y_0, \ldots, y_{N-1} \rangle$ of moves by Player II, and consider the play of a game $G_N(A)$ which arises as a result of this strategy being applied against these moves. We denote this play by $\sigma * t$. By symmetry, if τ is a strategy for Player II and $s = \langle x_0, \ldots, x_{N-1} \rangle$ the sequence of the opponents' moves, we denote the result by $s * \tau$. Formally we can give an inductive definition:

1.4.2 Definition.

1. Let σ be a strategy for player I in the game $G_N(A)$. For any $t = \langle y_0, \ldots, y_{N-1} \rangle$ we define

$$\sigma * t := \langle x_0, y_0, x_1, y_1, \ldots, x_{N-1}, y_{N-1} \rangle$$

where the x_i are given by the following inductive definition:

- $x_0 := \sigma(\langle\rangle)$
- $x_{i+1} := \sigma(\langle x_0, y_0, x_1, y_1, \ldots, x_i, y_i \rangle)$

2. Let τ be a strategy for player II in the game $G_N(A)$. For any $s = \langle x_0, \ldots, x_{N-1} \rangle$ we define

$$s * \tau := \langle x_0, y_0, x_1, y_1, \ldots, x_{N-1}, y_{N-1} \rangle$$

where the y_i are given by the following inductive definition:

- $y_0 := \sigma(\langle x_0 \rangle)$
- $y_{i+1} := \sigma(\langle x_0, y_0, x_1, y_1, \ldots, x_i, y_i, x_{i+1} \rangle)$

So a game where I uses strategy σ and II plays $\langle y_0, \ldots, y_{N-1} \rangle$ would look like this:

I:	$\sigma(\langle\rangle)$		$\sigma(\langle \sigma(\langle\rangle), y_0 \rangle)$		$\sigma(\langle \sigma(\langle\rangle), y_0, \sigma(\langle \sigma(\langle\rangle), y_0 \rangle), y_1 \rangle)$	
II:		y_0		y_1		$\ldots$

1.4.3 Definition. Let $G_N(A)$ be a game and σ a strategy for Player I. We denote by

$$\mathsf{Plays}_N(\sigma) := \{\sigma * t : t \in \omega^N\}$$

the set of all possible plays in the game $G_N(A)$ in which I plays according to σ. Similarly,

$$\mathsf{Plays}_N(\tau) := \{s * \tau : s \in \omega^N\}$$

denotes the set of all possible plays in which II plays according to τ.

Now we introduce what may be called the most crucial concept of all game theory:

1.4.4 Definition. Let $G_N(A)$ be a finite game.

1. A strategy σ is a *winning strategy for Player I* if for any t: $\sigma * t \in A$.
2. A strategy τ is a *winning strategy for Player II* if for any s: $s * \tau \notin A$.

1.4.5 Lemma. *For any $G_N(A)$, Players I and II cannot both have winning strategies.*

Proof. Exercise 2. □

1.5 Determinacy of finite games

Having introduced the concept of winning strategies it is natural to ask the following question: is it always the case that either Player I or Player II has a winning strategy in a given game? We refer to this is the *determinacy* of a game.

1.5.1 Definition. A game $G_N(A)$ is called *determined* if either Player I or Player II has a winning strategy.

1.5.2 Theorem. *Every finite game $G_N(A)$ is determined.*

Proof. Let us analyze the concept of a winning strategy once more. On close inspection it becomes clear that Player I has a winning strategy in the game $G_N(A)$ if and only if the following holds:

- $\exists x_0 \forall y_0 \exists x_1 \forall y_1 \dots \exists x_{N-1} \forall y_{N-1} \ (\langle x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1}\rangle \in A)$

So suppose I does *not* have a winning strategy. Then

- $\neg(\exists x_0 \forall y_0 \exists x_1 \forall y_1 \dots \exists x_{N-1} \forall y_{N-1} \ (\langle x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1}\rangle \in A))$

By elementary duality of first order logic, this implies in sequence

- $\forall x_0 \neg(\forall y_0 \exists x_1 \forall y_1 \dots \exists x_{N-1} \forall y_{N-1} \ (\langle x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1}\rangle \in A))$
- $\forall x_0 \exists y_0 \neg(\exists x_1 \forall y_1 \dots \exists x_{N-1} \forall y_{N-1} \ (\langle x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1}\rangle \in A))$
- $\forall x_0 \exists y_0 \forall x_1 \neg(\forall y_1 \dots \exists x_{N-1} \forall y_{N-1} \ (\langle x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1}\rangle \in A))$
- $\forall x_0 \exists y_0 \forall x_1 \exists y_1 \neg(\dots \exists x_{N-1} \forall y_{N-1} \ (\langle x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1}\rangle \in A))$

  $\dots$
- $\forall x_0 \exists y_0 \forall x_1 \exists y_1 \dots \forall x_{N-1} \exists y_{N-1} \ (\langle x_0, y_0, x_1, y_1, \dots, x_{N-1}, y_{N-1}\rangle \notin A)$

Now it is easy to see that the last statement holds if and only if Player II has a winning strategy in $G_N(A)$. □

For those not so fond of first-order logic, alternative proofs can easily be found: for example, using a backward induction argument on game-trees, or using the notion of a *defensive* strategy which we will introduce in Section 3.1.

For the remainder of this chapter we briefly discuss the implications of the above determinacy result for actual games. In our formalization of chess, it follows that either White has a winning strategy or Black has a strategy to win or draw. Of course, it now seems unsatisfactory to equate a draw to a win by Black, so this is what we can do to avoid it: simply define two different games, call them "white-chess" and "black-chess", which are played exactly as chess but in the first case, a draw is considered a win for white and in the second case, a win for black. Both games are finite and determined, so there are four possible combinations of assigning winning strategies to the two players. The following table illustates this and sums up the conclusion for real chess in each case ("w.s." abbreviates winning strategy):

White-chess	Black-chess	Real chess
White has a w.s.	White has a w.s.	White has a strategy to win
Black has a w.s.	White has a w.s.	Impossible
White has a w.s.	Black has a w.s.	Both White and Black have a strategy to draw
Black has a w.s.	Black has a w.s.	Black has a strategy to win

This leads to the following corollary of Theorem 1.5.2, typically credited to Ernst Zermelo in the 1913 paper [Zermelo 1913][1]

1.5.3 Corollary. *In chess, either White has a winning strategy, or Black has a winning strategy, or both have a drawing strategy.*

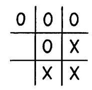
Note that, of course, the above corollary only tells us a mathematical fact, namely that *there is* such or such a strategy, and obviously does not tell us *which one it is!* That would, in effect, amount to the game of chess having been "solved" and having lost its character of being an actual game. To accomplish this, one would need to parse through the tree of all possible games of chess, a feat which would involve such enormously large numbers that it is practically impossible (although there are easier games than chess that have been "solved" in this sense). Moreover, such a tree-parsing method is only possible in games with a finite number of possibile moves (such as chess) but Theorem 1.5.2 applies equally well to finite games with an infinite possibility of moves.

[1] Although it is not entirely clear what Zermelo actually proved in this paper. See [Schwalbe & Walker 2001] for a discussion.

1.6 Exercises

1. Describe at least two ways of formalizing the game "tic-tac-toe" (noughts and crosses). What is the length of the game? What kind of winning, losing or drawing strategies do the players have?

2. Prove Lemma 1.4.5. (Hint: let $\sigma * \tau$ be the result of playing σ against τ, and see what happens.)



2 Infinite games

2.1 Basic definitions

We now extend our basic setting from finite to infinite games. This necessarily involves slightly more advanced mathematics and set theory, since the study of infinite objects is conceptually one level higher than that of finite objects. On the other hand, the formalism we developed for finite games was intentionally presented in such a way as to make the transition to infinite games smooth and straightforward.

As we shall now deal with infinite sequences of natural numbers rather than finite ones, our basic underlying object is the space of all functions from ω to ω, denoted by ω^ω (see introduction for all the relevant notation). Games of infinite length will produce infinite sequences, i.e., elements of ω^ω, and a pay-off set will now be a subset of ω^ω rather than a subset of ω^{2N}. For the rest, not much changes.

2.1.1 Definition. (Two-person, perfect-information infinite game.) Let A be an arbitrary subset of ω^ω. The game $G(A)$ is played as follows:

- There are two players, Player I and Player II, which take turns in picking one natural number at each step of the game.
- At each turn i, we denote Player I's choice by x_i and Player II's choice by y_i.
- In the limit, an infinite game "has been played", which looks as follows:

I:	x_0		x_1		$\dots$	
II:		y_0		y_1		$\dots$

  Now let $z := \langle x_0, y_0, x_1, y_1, x_2, y_2, \dots \rangle$ be an infinite sequence, called a *play of the game* $G(A)$. Formally, we should define

$$z(i) := \begin{cases} x_{i/2} & \text{if } i \text{ is even} \\ y_{(i-1)/2} & \text{if } i \text{ is odd} \end{cases}$$

- Player I *wins the game* $G(A)$ if $z \in A$, and Player II wins if $z \notin A$. The set A is called the *pay-off set* for Player I or the *set of winning conditions* for Player I.

Clearly, only the set A matters in the study of the game, and indeed the study of infinite games is closely related to the study of various types of sets $A \subseteq \omega^\omega$. This will become much more clear in Chapter 4 where we introduce a topology on the space ω^ω which we closely related to the standard topology on the real number continuum.

2.1.2 Definition. Let $G(A)$ be an infinite game. A *strategy for Player I* is a function

$$\sigma : \{s \in \omega^{<\omega} \ : \ |s| \text{ is even } \} \longrightarrow \omega$$

A *strategy for Player II* is a function

$$\tau : \{s \in \omega^{<\omega} \ : \ |s| \text{ is odd } \} \longrightarrow \omega$$

Given a strategy σ for Player I and an infinite sequence $y = \langle y_0, y_1, y_2 \dots \rangle$ of responses by Player II, we again use the notation $\sigma * y$ to denote the resultant play. Similarly, $x * \tau$ stands for the play resulting from Player I playing x and II playing according to strategy τ. Or, formally:

2.1.3 Definition.

1. Let σ be a strategy for player I. For any $y = \langle y_0, y_1, \dots \rangle$ define

   $$\sigma * y := \langle x_0, y_0, x_1, y_1, \dots \rangle$$

   where the x_i are given by the following inductive definition:

   - $x_0 := \sigma(\langle\rangle)$
   - $x_{i+1} := \sigma(\langle x_0, y_0, x_1, y_1, \dots, x_i, y_i \rangle)$

2. Let τ be a strategy for player II. For any $x = \langle x_0, x_1, \dots \rangle$ define

   $$s * \tau := \langle x_0, y_0, x_1, y_1, \dots \rangle$$

   where the y_i are given by the following inductive definition:

   - $y_0 := \sigma(\langle x_0 \rangle)$
   - $y_{i+1} := \sigma(\langle x_0, y_0, x_1, y_1, \dots, x_i, y_i, x_{i+1} \rangle)$

2.1.4 Definition. Let σ a strategy for Player I. We denote by

$$\mathsf{Plays}(\sigma) := \{\sigma * x : x \in \omega^\omega\}$$

the set of all possible infinite plays in which I plays according to σ. Similarly,

$$\mathsf{Plays}(\tau) := \{y * \tau : y \in \omega^\omega\}$$

is the set of all possible infinite plays in which II plays according to τ.

Just as in the finite case, we have the all-important concept of a winning strategy:

2.1.5 Definition. Let $A \subseteq \omega^\omega$ be a given pay-off set.

1. A strategy σ is a *winning strategy for Player I in* $G(A)$ if for any $y \in \omega^\omega$: $\sigma * y \in A$.
2. A strategy τ is a *winning strategy for Player II in* $G(A)$ if for any $x \in \omega^\omega$: $x * \tau \notin A$.

2.1.6 Lemma. *For any $A \subseteq \omega^\omega$, Players I and II cannot both have winning strategies.*

Proof. Exercise 1. □

2.2 Some examples

Consider the following game: Players I and II pick natural numbers and the first player to play a 5 loses. If no 5's have been played at all, Player I wins. How do we model this game? The only variable we have is the pay-off set A. In this game, you can see that the set is

$$A := \{z \in \omega^\omega : \forall n\ (z(n) \neq 5)\ \vee\ \exists n\ (z(2n+1) = 5)\}$$

Clearly Player I has an easy winning strategy in this game: don't play any 5's.

Another example: Players I and II pick numbers with the condition that each number has to be higher than the previous one. The first player to break that rule loses. If no-one has broken the rule, Player II wins. Here the pay-off set A of winning conditions for Player I is given by

$$A := \{z \in \omega^\omega : \exists n\ (z(2n+1) \leq z(2n))\}$$

Here II has a winning strategy: always play a number higher than the previous one played.

Both games above are based on essentially finitary rules, but since we have infinite games, we can easily introduce infinitary rules. Consider the following game: I and II pick numbers, and Player I wins if and only if he plays infinitely many 0's. This is modeled by

$$A := \{z \in \omega^\omega : \forall n \exists m \geq n\ (z(2m) = 0)\}$$

Although the rule is infinitary in the sense that the players will only know who won after infinitely many moves have been taken, it is nevertheless easy to see that Player I has a winning strategy, for example: play a 0 every time.

2.3 Cardinality arguments

Before discussing the determinacy of infinite games, let us prove some easy results about infinite games and strategies. As we mentioned before, the study of infinite games $G(A)$ is essentially related to the sets A themselves, and, in particular, their cardinalities.

2.3.1 Theorem. *Let $A \subseteq \omega^\omega$ be a countable set. Then II has a winning strategy in $G(A)$.*

Proof. Let A be enumerated by $\{a_0, a_1, a_2, \dots\}$. We describe a winning strategy τ for Player II. It is simply the following strategy: at your i-th move, play any natural number different from $a_i(2i+1)$ (this is the $(2i+1)$-st digit of the i-th element of A), regardless of the previous sequence of moves. Or, formally, define for all i:

$$\tau(\langle x_0, y_0, \dots, x_i \rangle) := a_i(2i+1) + 1$$

Let z be the result of this strategy against anything played by Player I, i.e., let $z = x * \tau$, for any $x \in \omega^\omega$. Write $z := \langle x_0, y_0, x_1, y_1, \dots \rangle$. By construction, for each natural i:

$$z(2i+1) = y_i \neq a_i(2i+1)$$

Hence, for each i:

$$z \neq a_i$$

But then $z \notin A$, proving that τ is indeed a winning strategy for Player II. □

We continue with some more "cardinality arguments". Recall that ω^ω has the cardinality of the continuum, i.e., $|\omega^\omega| = 2^{\aleph_0}$ (this is the same as the cardinality of the real numbers $\mathbb{R}$, $\mathscr{P}(\omega)$, $2^\omega := \{f : \omega \to \{0,1\}\}$ and so on.)

2.3.2 Definition. For strategies σ and τ of Player I resp. II, let f_σ and g_τ be functions from ω^ω to ω^ω defined by:

$$f_\sigma(y) := \sigma * y$$

$$g_\tau(x) := x * \tau$$

2.3.3 Lemma. *Every function f_σ is a bijection between ω^ω and $\mathsf{Plays}(\sigma)$. Every function g_τ is a bijection between ω^ω and $\mathsf{Plays}(\tau)$.*

Proof. Exercise 3. □

2.3.4 Corollary. *Let A be a set with $|A| < 2^{\aleph_0}$. Then Player I cannot have a winning strategy in $G(A)$.*[2]

[2]Note that if the Continuum Hypothesis is true, i.e., if $2^{\aleph_0}$ is the smallest uncountable cardinality, then by the previous Theorem 2.3.1 A must be countable so I must have a winning strategy. So this Corollary has relevance only if the Continuum Hypothesis is false.

Proof. If σ were a winning strategy for Player I, then by definition we would have $\mathsf{Plays}(\sigma) \subseteq A$. By Lemma 2.3.3 there is an injection from ω^ω to $\mathsf{Plays}(\sigma)$, namely f_σ. But then the cardinality of A must be at least that of ω^ω, namely $2^{\aleph_0}$. $\square$

Obviously these two theorems also hold with the roles of I and II reversed, i.e., if $\omega^\omega - A$ is countable then I has a winning strategy, and if $|\omega^\omega - A| < 2^{\aleph_0}$ then II cannot have a winning strategy.

2.4 Determinacy of infinite games

If we extend the definition from section 1.5, we get the following:

2.4.1 Definition. A game $G(A)$ is *determined* if either Player I or Player II has a winning strategy.

Since the game $G(A)$ depends only on the set A, we also say:

2.4.2 Definition. A set $A \subseteq \omega^\omega$ is *determined* if the game $G(A)$ is determined.

Is every game $G(A)$ determined? Note that we can no longer apply the "quantifier switch" of first-order logic as we did in the proof of Theorem 1.5.2, since we would now have to write an infinite sequence of alternating quantifiers $\exists x_0 \forall y_0 \exists x_1 \forall y_1 \dots$ which is not a valid logical formula. Indeed, we will now prove that using the full power of mathematics, to be precise the Axiom of Choice (which is one of the basic axioms of set theory, or foundations of mathematics), we can show that there are non-determined games. The proof will by necessity be more sophisticated then anything we have seen so far. In particular, it will be non-constructive, i.e., we will not produce a concrete game $G(A)$ and prove that it is not determined, but only prove that a non-determined game *must exist.*

The use of the Axiom of Choice is necessary here: one can show (using meta-mathematical arguments) that without using it, it is impossible to prove this result.

2.4.3 Theorem. *There exists a non-determined set, i.e., a set $A \subseteq \omega^\omega$ such that $G(A)$ is not determined.*

The proof of this theorem uses *transfinite induction on the ordinals* $\alpha < 2^{\aleph_0}$. However, since we do not assume familiarity with ordinals and cardinals in this course, we will supply a blackbox result which encompasses exactly what we need for the proof.

2.4.4 Definition. A *well-ordered* set is a set I with an order relation $\leq$ which is

1. Reflexive: $\forall \alpha \in I \ (\alpha \leq \alpha)$

2. Antisymmetric: $\forall \alpha, \beta \in I\ (\alpha \leq \beta \wedge \beta \leq \alpha \rightarrow \alpha = \beta)$
3. Transitive: $\forall \alpha, \beta, \gamma \in I\ (\alpha \leq \beta \wedge \beta \leq \gamma \rightarrow \alpha \leq \gamma)$
4. Linear: $\forall \alpha, \beta \in I\ (\alpha \leq \beta \vee \beta \leq \alpha)$
5. Well-founded: every subset $J \subseteq I$ contains an $\leq$-least element (i.e., an α such that $\forall \beta \in J\ (\alpha \leq \beta)$). Equivalently, it means that there is no infinite strictly descending sequence, i.e., no sequence $\{\alpha_i \in I : i \in \omega\}$ s.t.

$$\alpha_0 > \alpha_1 > \alpha_2 > \dots$$

If $(I, \leq)$ is a well-ordered set we can apply transfinite induction to that set.

2.4.5 Lemma. *For every set X, there exists a well-ordered set $(I, \leq)$, which we call the* index set for X*, such that*

1. $|I| = |X|$*, and*
2. *For every $\alpha \in I$, the set $\{\beta \in I : \beta < \alpha\}$ has cardinality strictly less than $|I| = |X|$.*

Proof. (ordinal and cardinal theory) By the Axiom of Choice every set X can be well-ordered, hence there is an ordinal α order-isomorphic to it. Let κ be $|\alpha| = |X|$, i.e., the least ordinal in bijection with α. Since κ is a cardinal, clearly $|\kappa| = \kappa = |\alpha| = |X|$, and for any $\gamma < \kappa$, the set $\{\beta < \kappa : \beta < \gamma\} = \beta$ has cardinality $< \kappa$. So $(\kappa, \in)$ is the desired index set. □

Those unfamiliar with ordinals can treat this lemma as a blackbox result and ignore its proof. Intuitively, one can compare the situation with that of a countable set X, in which case the index set is simply $(\omega, \leq)$ (the standard ordering of the natural numbers).

Proof of Theorem 2.4.3. We start by counting the possible number of strategies. A strategy is a function from a subset of $\omega^{<\omega}$ to ω. But it is easy to see that $\omega^{<\omega}$ is a countable set, and therefore can be identified with ω via some bijection. Hence, each strategy can be identified with a function from ω to ω. Therefore, there are exactly as many strategies as functions from ω to ω, namely $2^{\aleph_0}$.

Now let $\mathsf{STRAT\text{-}I}$ be the set of all possible strategies of Player I and apply Lemma 2.4.5 to that set. The index set I has cardinality $2^{\aleph_0}$, and the bijection between $\mathsf{STRAT\text{-}I}$ and I allows us to identify every strategy for Player I with an index $\alpha \in I$. Thus we can write

$$\mathsf{STRAT\text{-}I} = \{\sigma_\alpha \mid \alpha \in I\}$$

We can do the same thing for the set $\mathsf{STRAT\text{-}II}$ of strategies of Player II, and moreover we can use the same index set (since the cardinality is the same). So we also have

$$\mathsf{STRAT\text{-}II} = \{\tau_\alpha \mid \alpha \in I\}$$

Now we are going to produce two sets $A, B \subseteq \omega^\omega$ by induction on $(I, \leq)$. We are going to define

$$A := \{a_\alpha : \alpha \in I\}$$
$$B := \{b_\alpha : \alpha \in I\}$$

by the following simultaneous induction.

- **Base case:** Let $0 \in I$ stand for the $\leq$-least member of I. Arbitrarily pick any $a_0 \in \mathsf{Plays}(\tau_0)$. Now, $\mathsf{Plays}(\sigma_0)$ clearly contains more than one element, so we can pick $b_0 \in \mathsf{Plays}(\sigma_0)$ such that $b_0 \neq a_0$.

- **Induction step:** Let $\alpha \in I$ and suppose that for all $\beta < \alpha$, a_β and b_β have already been chosen. We will chose a_α and b_α.

  Note that since $\{b_\beta : \beta < \alpha\}$ is in bijection with $\{\beta \in I : \beta < \alpha\}$, it has cardinality strictly less than $2^{\aleph_0}$ (by Lemma 2.4.5 (2)). On the other hand, we already saw that $\mathsf{Plays}(\tau_\alpha)$ has cardinality $2^{\aleph_0}$. Therefore there is at least one element in $\mathsf{Plays}(\tau_\alpha)$ but not in $\{b_\beta : \beta < \alpha\}$. Pick any one of these and call it a_α.

  Now do the same for the collection $\{a_\beta : \beta < \alpha\} \cup \{a_\alpha\}$. This still has cardinality less than $2^{\aleph_0}$ whereas $\mathsf{Plays}(\sigma_\alpha)$ has cardinality $2^{\aleph_0}$, so we can pick a b_α in $\mathsf{Plays}(\sigma_\alpha)$ which is not a member of $\{a_\beta : \beta < \alpha\} \cup \{a_\alpha\}$.

After having completed the inductive definition, we have defined our two sets A and B.

Claim 1. *$A \cap B = \varnothing$.*

Proof of Claim 1. Take any $a \in A$. By construction, there is some $\alpha \in I$ such that $a = a_\alpha$. Now, recall that at "stage α" of the inductive procedure, we made sure that a_α is not equal to b_β for any $\beta < \alpha$. On the other hand, at each "stage γ" for $\gamma \geq \alpha$, we made sure that b_γ is not equal to a_α. Hence a_α is not equal to any $b \in B$, proving the claim. $\square$

Claim 2. *A is not determined.*

Proof of Claim 2. First, assume that I has a winning strategy σ in $G(A)$. Then $\mathsf{Plays}(\sigma) \subseteq A$. But there is an $\alpha \in I$ such that $\sigma = \sigma_\alpha$. At "stage α" of the inductive procedure we picked a $b_\alpha \in \mathsf{Plays}(\sigma_\alpha)$. But by Claim 1, b_α cannot be in A—contradiction.

Now assume II has a winning strategy τ in $G(A)$. Then $\mathsf{Plays}(\tau) \cap A = \varnothing$. Again, $\tau = \tau_\alpha$ for some α, but at "stage α" we picked $a_\alpha \in \mathsf{Plays}(\tau_\alpha)$—contradiction. $\square$

This completes the proof of the theorem. $\square$

Although this is a delimitative result showing that we cannot hope to prove that *all* games are determined, we will go on to show that this does not undermine the whole enterprise of infinite game theory. In fact, one may argue that for all sets A which are *sufficiently interesting* (i.e., not too non-constructive), determinacy holds. The core of this idea lies in the Gale-Stewart Theorem which we treat in the next chapter.

2.5 Exercises

1. Prove Lemma 2.1.6.

2. Let $z := \langle z(0), z(1), z(2), \dots \rangle$ be an infinite sequence. Describe informally the game $G(A)$ where $A = \{z\}$. Who has a winning strategy in this game? How many moves does that player need to make sure he or she has won the game?

3. Prove Lemma 2.3.3.

4. For every set $A \subseteq \omega^\omega$ and every $n \in \omega$, define

$$\langle n \rangle ^\frown \overline{A} := \{\langle n \rangle ^\frown x : x \notin A\}$$

   (a) Prove, or at least argue informally, that for every $A \subseteq \omega^\omega$, Player II has a winning strategy in $G(A)$ if and only if for every n, Player I has a winning strategy in $G(\langle n \rangle ^\frown \overline{A})$.

   (b) Similarly, prove that for every $A \subseteq \omega^\omega$, Player I has a winning strategy in $G(A)$ if and only if there is some n such that Player II has a winning strategy in $G(\langle n \rangle ^\frown \overline{A})$.

5.* Adapt the proof of Theorem 2.4.3 to prove that the property of "being determined" is not closed under complements, i.e., that there is a set A such that $G(A)$ is determined but $G(\omega^\omega - A)$ is not determined.

3 The Gale-Stewart Theorem

3.1 Defensive strategies

Imagine the following scenario: in a particular game (finite or infinite) Player I does not have a winning strategy. Will that always remain the case at each position of the game? In other words, after n moves have been played, will it still be the case that Player I has no winning strategy in the game *from the n-th move onwards*? Surely, this doesn't seem right. After all, Player II might make a mistake. She might play badly, meaning that even though I had no winning strategy to begin with, he might acquire one following a mistake made by Player II. Using the concept of a *defensive strategy*, we will now prove that if the opponent plays "correctly", such a situation will never happen. The defensive strategy is essentially the strategy of "not making any mistakes".

To formulate this precisely, we need to specify what we mean by a certain *position* in a game.

3.1.1 Definition. Let $G(A)$ be an infinite game. If s is a finite sequence of even length, then $G(A; s)$ denotes the game in which Player I starts by playing x_0, Player II continues with y_0, etc., and Player I wins the game $G(A; s)$ if and only if $s^\frown \langle x_0, y_0, x_1, y_1, \dots \rangle \in A$.

So $G(A; s)$ refers to the game $G(A)$ but instead of starting at the initial position, starting at position s, i.e., from the position in which the first moves played are exactly $s(0), s(1), \dots, s(n-1)$ (where $n = |s|$). The reason we only consider sequences of even length is because it corresponds to a certain number of complete moves having been made (and it is again I's turn to move).

3.1.2 Lemma. *The game $G(A; s)$ is exactly the same as the game $G(A/s)$, where A/s is the set defined by*

$$A/s := \{x \in \omega^\omega : s^\frown x \in A\}$$

Proof. Exercise 1. □

Because of this Lemma, when talking about games at certain positions, we do not really need to introduce new terminology but can simply refer to a different game. Thus, I has a winning strategy in $G(A; s)$ if and only if he has one in $G(A/s)$, and the same holds for Player II.

Concerning positions in a game, we also use the following notation: if t is a finite sequence of length n and σ is a strategy for Player I, $\sigma * t$ is the position in the game which results when I plays according to σ and II plays the sequence t, for the first n moves of the game. So $\sigma * t$ is a sequence of length $2n$. Similarly, if τ is a strategy for II and s a sequence of length n, we denote the result by $s * \tau$, which is also a sequence of length $2n$. We will not give the formal definition since this is a straightforward analogy to Definition 2.1.3

Now we can give the definition of defensive strategies.

3.1.3 Definition. Fix a game $G(A)$.

1. A strategy ∂_I is *defensive for Player I* if for all t, Player II does not have a winning strategy in $G(A, \partial_I * t)$.
2. A strategy ∂_{II} is *defensive for Player II* if for all s, Player I does not have a winning strategy in $G(A, s * \partial_{II})$.

It is not clear that such defensive strategies even exist. Clearly, if one Player has a winning strategy in $G(A)$ then his or her opponent cannot have a defensive strategy (an empty sequence $\langle\rangle$ would be a counterexample). However, we will prove inductively that the converse is true.

3.1.4 Theorem. *Let $G(A)$ be an infinite game.*

1. *If Player II does not have a winning strategy, then Player I has a defensive strategy.*
2. *If Player I does not have a winning strategy, then Player II has a defensive strategy.*

Proof. We construct defensive strategies inductively. The idea for both parts is exactly the same, but we will give both proofs since the details are slightly different.

1. The idea is to define ∂_I such that for any t, Player II does not have a winning strategy in $G(A, \partial_I * t)$, by induction on the length of t. The base case is $t = \langle\rangle$. In that case we don't need to specify ∂_I since in any case $\partial_I * t$ is also the empty sequence, i.e., the initial position of the game, and Player II does not have a winning strategy in $G(A)$ by assumption.

   Next, we assume that for all t of length n, Player II does not have a winning strategy in $G(A; \partial_I * t)$. Fix a t and for convenience set $p := \partial_I * t$.

   Claim. *There is an x_0 such that for all y_0, Player II still does not have a winning strategy in $G(A; p^\frown \langle x_0, y_0\rangle)$.*

   Proof. If not, then for each x_0 there exists a corresponding y_0 such that Player II has a winning strategy, say τ_{x_0}, in the game $G(A; p^\frown \langle x_0, y_0\rangle)$. But then Player II already had a winning strategy in the game $G(A; p)$, namely the following one: if I plays x_0, reply with y_0 and then continue following strategy τ_{x_0}. This contradicts our inductive hypothesis. □

   Now we extend ∂_I and define $\partial_I(p) := x_0$, for that particular x_0 as in the Claim. No matter which y_0 II plays in return, she will not gain a winning strategy in the game $G(A; p^\frown \langle \partial_I(p), y_0\rangle)$. Since the proof works for arbitrary p, we see that we have extended the induction hypothesis with one more step, namely, for all t of length $n + 1$, II does not have a winning strategy in $G(A, \partial_I * t)$.

2. Here we do the same with the roles of I and II reversed, i.e., we inductively construct ∂_{II} in such a way that for all s, Player I does not have a winning strategy in $G(A; s * \partial_{II})$. If $s = \langle\rangle$ then again $G(A; s * \partial_{II})$ is just $G(A)$ and I has no winning strategy by assumption.

   Now assume for all s of length n, Player I does not have a winning strategy in $G(A; s * \partial_{II})$. Set $p := s * \partial_{II}$.

   Claim. *For all x_0 there is a y_0 such that Player I still does not have a winning strategy in $G(A; p^\frown \langle x_0, y_0\rangle)$.*

   Proof. If not, then there is an x_0 such that for any y_0, Player I has a winning strategy σ_{x_0} in $G(A; p^\frown \langle x_0, y_0\rangle)$. But then I already had a winning strategy in $G(A; p)$, namely the following one: play x_0, and after any reply y_0 continue following σ_{x_0}. This contradicts our inductive hypothesis. □

   Now extend ∂_{II} by stipulating that for any x_0, $\partial_{II}(p^\frown \langle x_0\rangle) := y_0$, where y_0 depends on x_0 as in the Claim. Since the proof worked for arbitrary p, we have again extended the induction hypothesis with one more step, namely, for all s of length $n + 1$, II does not have a winning strategy in $G(A, s * \partial_{II})$.

□

Of course, following a defensive strategy does not guarantee a win. It might, in theory, happen that even though Player I does not have a winning strategy at any finite position of the game, he still wins the game (because of the complex way the pay-off set is constructed). In the next section we show that this does not happen at least for games with certain kinds of pay-off sets.

3.2 Finitely decidable sets

In section 2.2 we have seen some examples of *finitary* rules (intuitively, those satisfied or broken at some finite stage of the game), and *infinitary* ones (those requiring the entire infinite length of the game to verify). We make this precise in the following definition regarding subsets of ω^ω. Recall that $\lhd$ denotes *initial segments*.

3.2.1 Definition. A set $A \subseteq \omega^\omega$ is *finitely decidable* if for all $x \in \omega^\omega$:

$$x \in A \implies \exists s \lhd x \, \forall y \, (s \lhd y \rightarrow y \in A)$$

In words, this definition says that if x is in A then there is a finite initial segment s of x such that any other infinite extension of s is in A as well.

You may also think about this as follows: suppose you are given an infinite sequence x and you go along its values: $x(0), x(1), x(2)$ etc. and constantly ask the question: is the infinite x going to be a member of A, or not? In general, you may only know the answer to this question in infinitely many steps, i.e., after you have considered all the $x(n)$'s. However, if A is finitely decidable, then this is not the case: at some finite stage, you will know whether x is in A or not. The membership of x in A is decided at a finite stage—hence the name "finitely decidable".

The following theorem, one of the crucial early results of infinite game theory, is known by the name "Gale-Stewart Theorem", referring to the paper [Gale & Stewart 1953].

3.2.2 Theorem. (Gale-Stewart, 1953) *If A is finitely decidable then $G(A)$ is determined.*

Proof. Suppose I does not have a winning strategy in the game $G(A)$. Then by Theorem 3.1.4 Player II has a defensive strategy ∂_{II}. We claim that ∂_{II} is in fact a winning strategy for Player II. Take any $x \in \omega^\omega$. We must show that $x * \partial_{II} \notin A$. Well, if $x * \partial_{II} \in A$ then by finite decidability there is an $s \lhd x * \partial_{II}$ such that all y with $s \lhd y$ are in A. But then Player I has a trivial winning strategy in the game $G(A; s)$—play any number whatsoever. Since s is an initial segment of $x * \partial_{II}$, it is a position in which II plays according to ∂_{II}. But then I having a winning strategy in $G(A; s)$ contradicts the definition of a defensive strategy! Hence we conclude that $x * \partial_{II} \notin A$ and this completes the proof. □

An analogous proof, with the roles of Player I and II reversed, shows the following:

3.2.3 Theorem. *If $\omega^\omega - A$ is finitely decidable then $G(A)$ is determined.*

Proof. Exercise 2. □

3.3 Exercises

1. Prove Lemma 3.1.2.
2. Prove Theorem 3.2.3.
3. Let $x \in \omega^\omega$. Is $\{x\}$ a finitely decidable set? Is $G(\{x\})$ determined?
4. Let $G_N(A)$ be a finite game of length N. Reformulate this game as an infinite game and prove that it is determined using the Gale-Stewart theorem.

4 Topology on the Baire space

4.1 Basic concepts

In this chapter, we will delve much deeper into the underlying structure of the set ω^ω of infinite sequences of natural numbers. By introducing a topology (and metric) on ω^ω, we will be able to classify sets $A \subseteq \omega^\omega$ better and the connection with the Gale-Stewart theorem will become more apparent.

From now on, we will assume some knowledge of basic topology, although we hope to present the theory in such a way that even readers not familiar with general topology will be able to understand the specific case of the topology on ω^ω.

4.1.1 Definition. Let $s \in \omega^{<\omega}$ be a finite sequence.

- We define
$$O(s) := \{x \in \omega^\omega : s \lhd x\}$$
The sets $O(s)$ are called *basic open*.

- A set $A \subseteq \omega^\omega$ is called *open* if it is a union of basic open sets, i.e., if $A = \bigcup\{O(s) : s \in J\}$ for some subset $J \subseteq \omega^{<\omega}$ (note that the empty set $\varnothing$ is open since it is a vacuous union of basic open sets).

- A set $A \subseteq \omega^\omega$ is called *closed* if its complement $\omega^\omega - A$ is open.

Recall that a space together with a collection of "open" subsets forms a *topological space* if the following conditions are satisfied:

1. $\varnothing$ and the entire space are open,
2. an arbitrary union of open sets is open, and
3. a finite intersection of open sets if open.

We will now verify that ω^ω with the collection of open sets as defined above forms a topological space.

4.1.2 Definition. Let $s, t \in \omega^{<\omega}$. We say that s and t are *compatible*, notation $s||t$, if either $s \lhd t$ or $t \lhd s$ (or $s = t$). Otherwise s and t are called *incompatible*, denoted by $s \perp t$.

4.1.3 Lemma. *Let $s, t \in \omega^{<\omega}$. The following holds:*

1. *$s \lhd t$ if and only if $O(t) \subseteq O(s)$,*
2. *$s||t$ if and only if $O(s) \subseteq O(t)$ or $O(t) \subseteq O(s)$,*
3. *$s \perp t$ if and only if $O(s) \cap O(t) = \varnothing$,*
4. *$O(s) \cap O(t)$ is either $\varnothing$ or basic open.*

Proof.

1. Suppose $s \lhd t$. Then $x \in O(t) \to t \lhd x \to s \lhd x \to x \in O(s)$. Conversely, suppose $s \ntriangleleft t$. Then either $|t| < |s|$ or there exists an i such that $s(i) \neq t(i)$. In either case, we can define an extension x of t such that for some i, $x(i) \neq s(i)$ which shows that $x \in O(t)$ but $x \notin O(s)$.

2. This follows directly from point 1.

3. If $s \perp t$ then there is an i such that $s(i) \neq t(i)$. Then any extension of x is not an extension of t, hence $O(s) \cap O(t) = \varnothing$.

4. Either $s||t$ or $s \perp t$. Hence $O(s) \cap O(t)$ is either $O(s)$, or $O(t)$, or $\varnothing$.

□

4.1.4 Corollary. *The space ω^ω with the collection of open sets is a topological space.*

Proof. We verify the three required conditions:

1. $\varnothing$ is clearly open, and ω^ω is open because it is equal to $O(\langle\rangle)$.

2. A union of open sets is open since a union of unions of basic open sets is itself a union of basic open sets.

3. Let A, B be open, and we must verify that $A \cap B$ is open. But if $A = \bigcup\{O(s) : s \in J_1\}$ and $B = \bigcup\{O(t) : t \in J_2\}$ then $A \cap B = \bigcup\{O(s) \cap O(t) : s \in J_1, t \in J_2\}$ and each $O(s) \cap O(t)$ is either basic open or $\varnothing$ by Lemma 4.1.3 (4), hence $A \cap B$ is open. □

This topological space is called *Baire space*. It has many similarities with the real line $\mathbb{R}$, in fact so many that pure set theorists prefer to study the Baire space instead of $\mathbb{R}$, and call elements $x \in \omega^\omega$ *real numbers*. The Baire space is homeomorphic to the space of irrational numbers $\mathbb{R} - \mathbb{Q}$ with the standard topology. But there is also an important difference: the Baire space is *totally disconnected*, which follows from the following Lemma.

4.1.5 Lemma. *For every s, $O(s)$ is clopen (closed and open).*

Proof. Let $n = |s|$ and we claim that $\omega^\omega - O(s) = \bigcup\{O(t) : |t| = n \text{ and } t \neq s\}$. If $x \notin O(s)$ then $s \ntriangleleft x$, so let $t := x \restriction n$. Clearly $|t| = n$, $t \neq s$ and $t \lhd x$, so $x \in O(t)$. Conversely, if $x \in O(s)$ then $s \lhd x$ so for any t of length n and $t \neq s$, clearly $t \ntriangleleft x$. Hence $x \notin O(t)$ for any such t. □

We can also consider ω^ω as a *metric space* and define the following metric d to measure distance between two points x and y.

4.1.6 Definition. The metric on Baire space is the function $d : \omega^\omega \times \omega^\omega \longrightarrow \mathbb{R}$ given by

$$d(x, y) := \begin{cases} 0 & \text{if } x = y \\ 1/2^n & \text{where } n \text{ is least s.t. } x(n) \neq y(n) \end{cases}$$

It is not hard to verify that the standard axioms of a metric space apply. Moreover, the topology induced by this metric is the same as the topology described above, as follow from the following Lemma.

4.1.7 Lemma. *For any x and n, $O(x \upharpoonright n)$ is the open ball around x with radius $\epsilon = 1/2^n$.*

Proof. Exercise 4. □

4.2 Convergence and continuity

4.2.1 Definition. An infinite sequence $\{x_n\}_{n\in\omega}$ is said to *converge to x*, notation "$x_n \to x$", if the following condition holds:

$$\forall s \lhd x \; \exists N \; \forall n \geq N \; (s \lhd x_n)$$

Such an x is called the *limit of* $\{x_n\}_{n\in\omega}$ and denoted by

$$x = \lim_{n\to\infty} x_n$$

If such x does not exist, we call the sequence *divergent*.

We can give two equivalent definitions of convergence:

1. $x_n \to x$ if and only if for all basic open neighbourhoods $O(s)$ of x, $\exists N \; \forall n \geq n \; (x_n \in O(s))$, and

2. $x_n \to x$ if and only if $\forall i \; \exists N \; \forall n \geq N \; x_n(i)$ is constant.

The first point shows that convergence in the sense of Definition 4.2.1 coincides with the normal definition of convergence in a topological space, whereas the second point shows that convergence of infinite functions is pointwise convergence.

Now let's turn to continuous functions from ω^ω to ω^ω. The standard definition is that a function from a topological space to another is *continuous* if the pre-image of open sets is open, i.e., if O is open then $f^{-1}[O]$ is open. In the Baire space, this is equivalent to saying that f preserves limits, i.e., if the sequence $\{x_n\}_{n\in\omega}$ converges to x then $\{f(x_n)\}_{n\in\omega}$ converges to $f(x)$. In other words

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right)$$

Using this we can define continuity of functions from ω^ω to ω^ω using the following direct clause:

4.2.2 Definition. A function $f : \omega^\omega \longrightarrow \omega^\omega$ is *continuous* if for each $x \in \omega^\omega$ we have

$$\forall s \lhd f(x)\ \exists t \lhd x\ \forall y\ (t \lhd y \to s \lhd f(y))$$

4.3 Trees

Now we introduce the notion of a tree. This section will be slightly more technical then the rest and, in principle, can be omitted without continuity, although we will return to trees in Chapter 5. The term "tree" has various meanings in mathematics, and we stress that here we only present the notion relevant to the topology on Baire space.

The idea is that since infinite sequences, or real numbers, $x \in \omega^\omega$ can be finitely approximated by its initial segments $s \lhd x$, a whole set $A \subseteq \omega^\omega$ can be approximated by the set of all initial segments of all its members. Although a real x is completely determined by the set of all its initial segments $\{s \in \omega^{<\omega} : s \lhd x\}$, this does not apply to sets of real numbers. One simple reason is that there are $2^{(2^{\aleph_0})}$ possible subsets of ω^ω whereas there are only $2^{\aleph_0}$ possible sets of finite sequences (because $\omega^{<\omega}$ is countable). Hence, the transition from $A \subseteq \omega^\omega$ to the set of finite approximations of all its members can, in general, involve loss of information. But we will see that this is not the case for *closed* A, and that in fact there is a one-to-one correspondence between trees and closed sets.

4.3.1 Definition. A *tree* T is any collection of finite sequences $T \subseteq \omega^{<\omega}$ closed under initial segments:

$$s \in T \ \wedge\ t \lhd s \ \to\ t \in T$$

It is easy to see why such objects are called "trees": the finite sequences are like nodes that can branch off in different directions. Because of this we use standard terminology for our trees:

4.3.2 Definition. Let T be a tree.

1. A $t \in T$ is called a *node of* T. The *successors of* t are all nodes $s \in T$ such that $t \lhd s$ and $|s| = |t| + 1$.
2. A $t \in T$ is called *terminal* if it has no successors.
3. A $t \in T$ is called *non-splitting* if it has only one successor.
4. A $t \in T$ is called *finitely splitting* if it has only finitely many successors, and *infinitely splitting* otherwise.

4.3.3 Definition. For a tree T, a *branch through* T is any $x \in \omega^\omega$ such that $\forall s \lhd x : s \in T$. The set of branches through T is denoted by $[T]$.

4.3.4 Definition. Let $A \subseteq \omega^\omega$ be a set of real numbers. The *tree of* A, denoted by $T(A)$, is the tree of all initial segments of all members of A, i.e.,

$$T(A) := \{s \in \omega^{<\omega} : s \lhd x \text{ for some } x \in A\}$$

If $x \in A$ then any $s \lhd x$ is in $T(A)$, and hence x is a branch through $T(A)$, i.e., $A \subseteq [T(A)]$ holds. Does the reverse inclusion hold? As we already mentioned above, this cannot be the case in general, because there are more subsets of ω^ω than there are trees (i.e., the mapping $A \mapsto [T(A)]$ cannot be injective.) But it is much more instructive to see an explicit example of a set A which is not equal to $[T(A)]$.

Example. Let $A = \{x_n : n \in \omega\}$ where

$$x_n(i) := \begin{cases} 1 & \text{if } i < n \\ 0 & \text{otherwise} \end{cases}$$

So

$$\begin{aligned} x_0 &= \langle 0, 0, 0, \dots \rangle \\ x_1 &= \langle 1, 0, 0, 0, \dots \rangle \\ x_2 &= \langle 1, 1, 0, 0, 0, \dots \rangle \\ x_3 &= \langle 1, 1, 1, 0, 0, 0, \dots \rangle \\ &\text{etc} \dots \end{aligned}$$

Consider the real $x = \langle 1, 1, 1, 1, \dots \rangle$. If s is any initial segment of x of length n, then clearly $s \lhd x_n$, hence $s \in T(A)$. Therefore $x \in [T(A)]$. However, x was not on our list of x_n's, so $x \notin A$.

The reason this happens is because the set A is not topologically closed. But for closed A, there is a neat correspondence as the following Theorem shows.

4.3.5 Theorem.

1. *For any tree T, $[T]$ is a closed set.*
2. *For any closed set C, $[T(C)] = C$.*

Proof.

1. We will show that $\omega^\omega - [T] = \bigcup\{O(t) : t \notin T\}$. First let $x \notin [T]$. By definition, there is $t \lhd x$ such that $t \notin T$. But then $x \in O(t)$. Conversely, suppose $x \in O(t)$ for some $t \notin T$. Then $t \lhd x$. Again, by definition, $x \notin [T]$.
2. We already saw that $C \subseteq [T(C)]$ always holds, so it remains to prove the reverse inclusion. So suppose $x \in [T(C)]$, and towards contradiction assume $x \notin C$. Since the complement of C is open, x is contained in some $O(t)$ such that $O(t) \cap C = \varnothing$. But $x \in [T(C)]$ and $t \lhd x$, so it must be the case that $t \in T(C)$. But by definition this means that t is the initial segment of some real in C, i.e., there should be at least one $y \in C$ such that $t \lhd y$. But then $y \in O(t)$ and thus $O(t) \cap C \neq \varnothing$ which contradicts what we showed earlier. □

So there is a one-one correspondence between closed sets and trees, given by $C \mapsto T(C)$ one way and $T \mapsto [T]$ in the other. Moreover, for any set A, $[T(A)]$ is the smallest closed set containing A as a subset, since if C were any other closed set with $A \subseteq C$ we would have $[T(A)] \subseteq [T(C)] = C$. Therefore, the operation $A \mapsto [T(A)]$ is the *topological closure* of the set A.

Another way of thinking about this is the following: recall that in any metric space, a set C is closed if and only if for any sequence $\{x_n\}_{n\in\omega}$ which converges to x, if every $x_n \in C$ then $x \in C$—we say that C is *closed under limit points.* Using Definition 4.2.1 it is easy to see that $[T]$ is closed under limit points and that the operation $A \mapsto [T(A)]$ is exactly the closure under such limit points—any x which is a limit of a sequence $\{x_n\}_{n\in\omega}$ in A will be adjoined to the set by the transition from A to $[T(A)]$.

4.4 Topology and determinacy

One connection between topology and determinacy was already implicitly established in section 3.2, even though we had not mentioned closed or open sets back then. If you have not already guessed, the connection is quite trivial:

4.4.1 Lemma. *A set $A \subseteq \omega^\omega$ is open if and only if it is finitely decidable*

Proof. Let A be open and $x \in A$. By definition there must be a basic open $O(t) \subseteq A$ such that $x \in O(t)$. But then $t \lhd x$ and every y with $t \lhd y$ is in $O(t)$ and hence in A. Therefore A is finitely decidable.

Conversely, if A is finitely decidable then for every $x \in A$ there is a $t \lhd x$ such that any y with $t \lhd y$ is in A. This is the same as saying that for every x there exists a basic open neighbourhood $O(t)$ of x such that $O(t) \subseteq A$. But then A is open, since it is the union of all these basic open sets. □

4.4.2 Corollary. *If A is open or closed then $G(A)$ is determined.*

Proof. This follows directly from the above Lemma and Theorems 3.2.2 and 3.2.3. □

A natural next question is: what about the determinacy of games with more complex pay-off sets? If we allow countable unions of closed set, we get the so-called F_σ sets. Similarly, countable intersections of open sets are called G_δ sets. These are really more complex then open or closed, but still fairly simple. It is possible (and not so hard) to prove that these are also determined. But we will not bother with that because there exists a far stronger result which subsumes these cases.

4.4.3 Definition. A σ-algebra of subsets of ω^ω is a collection $\mathcal{A} \subseteq \mathscr{P}(\omega^\omega)$ closed under countable unions and complements, i.e.,

1. If $A_n \in \mathcal{A}$ for $n \in \omega$ then $\bigcup_n A_n \in \mathcal{A}$, and

2. If $A \in \mathcal{A}$ then $\omega^\omega - A \in \mathcal{A}$.

4.4.4 Definition. The *Borel σ-algebra $\mathcal{B}$* is the smallest σ-algebra containing all open sets. Members of $\mathcal{B}$ are called *Borel sets*.

4.4.5 Theorem. (Martin, 1975) *For every Borel set B, $G(B)$ is determined.*

Unfortunately, it is beyond the scope of this course to prove this theorem, which requires the use of more complex kinds of infinite games than we have studied so far. But the consequences of this theorem are quite broad and we shall see some applications in the next section.

The Borel sets form a large group of subsets of ω^ω, and for many applications in topology and analysis, they are all one cares about. Nevertheless, there are many non-Borel sets and we would also like to go beyond and look at classes extending the Borel σ-algebra. There are many such classes studied in topology and set theory, most notably the *projective pointclasses*, but also others. We make a general definition:

4.4.6 Definition. Let $\mathbf{\Gamma} \subseteq \mathscr{P}(\omega^\omega)$ be a collection of subsets of ω^ω. We call $\mathbf{\Gamma}$ a *boldface pointclass*[3] if it is closed under continuous pre-images and intersections with closed sets, i.e.,

1. For all A, if $A \in \mathbf{\Gamma}$ then $f^{-1}[A] \in \mathbf{\Gamma}$, and
2. For all closed sets C, if $A \in \mathbf{\Gamma}$ then $A \cap C \in \mathbf{\Gamma}$.

The closed, F_σ, G_δ sets, as well as the Borel sets $\mathcal{B}$, are examples of boldface pointclasses. $\mathscr{P}(\omega^\omega)$ is also a trivial boldface pointclass.

4.4.7 Definition. For a boldface pointclass $\mathbf{\Gamma}$, "$\mathsf{Det}(\mathbf{\Gamma})$" abbreviates the statement: "for every $A \in \mathbf{\Gamma}$, $G(A)$ is determined."

So the Gale-Stewart theorem says Det(open) and Det(closed), and Martin's Theorem 4.4.5 says $\mathsf{Det}(\mathcal{B})$. On the other hand, Theorem 2.4.3 says $\neg\mathsf{Det}(\mathscr{P}(\omega^\omega))$.

It turns out that if we focus on pointclasses $\mathbf{\Gamma}$ that extend the Borel sets, but are still far below $\mathscr{P}(\omega^\omega)$, then typically the statement $\mathsf{Det}(\mathbf{\Gamma})$ is independent of the basic axioms of set theory, i.e., it is not possible to prove or refute the statement based on the axioms alone. Nevertheless, as long as $\mathsf{Det}(\mathbf{\Gamma})$ is not outright contradictory, we can take it as an axiom and develop its consequences. The most typical instances is when $\mathbf{\Gamma}$ refers to classes in the *projective hierarchy* (the $\mathbf{\Sigma}^1_n$ and $\mathbf{\Delta}^1_n$ sets) or the class of all projective sets ($\bigcup_n \mathbf{\Sigma}^1_n$). For the latter case, the axiom is frequently referred to as PD (the axiom of Projective Determinacy) in the literature and is an axiom often considered in the context of large cardinals.

[3]The name comes from the fact that such classes were traditionally denoted by boldface letters. Although it is a funny blend of syntax and semantics which, formally speaking, is wrong and might lead to problems, in practice this does not occur because usually people have explicitly defined $\mathbf{\Gamma}$ in mind.

All of this lies far beyond the scope of our course. In the next section, we will look at arbitrary pointclasses $\mathbf{\Gamma}$, and see what happens if we assume $\mathsf{Det}(\mathbf{\Gamma})$ as an axiom.

4.5 Exercises

1. Prove that the metric d defined in Definition 4.1.6 satisfies the *triangle inequality* (one of the four metric-axioms):
$$d(x,z) \leq d(x,y) + d(y,z)$$
2. Prove that the Baire space is a Hausdorff space, i.e., that for any two $x \neq y$ there are two disjoint open neighbourhoods of x and y.
3. A topological space is called *totally separated* if for every two $x \neq y$ there are open sets U and V such that $x \in U$, $y \in V$ and $U \cup V$ equals the whole space. Prove that the Baire space is totally separated.
4. Prove Lemma 4.1.7
5. Prove that for any x, the set $T_{=x} := \{s \in \omega^{<\omega} : s \lhd x\}$ is a tree, that $[T_{=x}] = \{x\}$ and conclude that singletons are closed.
6. For $x \in \omega^\omega$ let $A_{\leq x} := \{y \in \omega^\omega : \forall n \ (y(n) \leq x(n))\}$ and $T_{\leq x} := \{s \in \omega^{<\omega} : \forall n \ (s(n) \leq x(n)\}$. Show that $T_{\leq x}$ is a tree, that $[T_{\leq x}] = A$, and conclude that $A_{\leq x}$ is closed.
7. Repeat the previous exercise for the sets $A_{\geq x}$ and $T_{\geq x}$ defined analogously but with "$\leq$" replaced by "$\geq$".
8. Prove that for any $n, m \in \omega$, the set $A_{n \mapsto m} := \{x \in \omega^\omega \mid x(n) = m\}$ is closed.
9. Let C_n be a closed set for every n. Show that the set $C := \{x \in \omega^\omega \mid \forall n \ (x \in C_n)\}$ is closed. Come up with an example showing that this does not hold for the set $A := \{x \in \omega^\omega \mid \exists n \ (x \in C_n)\}$.
10. Conclude from Exercises 8 and 9 (or prove directly) that for infinite sequences $\langle n_0, n_1, \dots \rangle$ and $\langle m_0, m_1, \dots \rangle$ the set $A_{\vec{n} \mapsto \vec{m}} := \{x \in \omega^\omega : \forall i \ (x(n_i) = m_i)\}$ is closed.
11. A tree is called *pruned* if every node has a successor, i.e., $\forall s \in T \ \exists t \in T$ such that $s \lhd t$. Show that every tree T can be turned into a pruned tree $\mathrm{pr}(T)$ in such a way that $[\mathrm{pr}(T)] = [T]$.
12. * A set K in a topological space is called *compact* if every infinite cover of K by open sets has a finite subcover, i.e., if for every J and $K \subseteq \bigcup\{O_j : j \in J\}$ with each O_j open, there exists a finite subset $I \subseteq J$ such that $K \subseteq \bigcup\{O_j : j \in I\}$. Show that in the Baire space, a closed set K is compact if and only if in the tree $T(K)$, every node is finitely splitting.

5 Applications of infinite games

5.1 Continuous coding

We have now arrived at the stage promised in the introduction, namely where we can construct and use infinite games as tools in the study of various mathematical objects. In the sections following this one, we will present four different mathematical topics related to the continuum (Baire space and/or real numbers), and in each case show how infinite games help us to understand these topics better. The general structure of all our theorems will be as follows: suppose $\mathbf{\Gamma}$ is a boldface pointclass, and Φ is some mathematical property of subsets of ω^ω or some other topological space representing the "continuum" that we are interested in for whatever reason. We will prove that *if* we assume $\mathsf{Det}(\mathbf{\Gamma})$, i.e., if all sets in $\mathbf{\Gamma}$ are determined, *then* we can conclude that all sets in $\mathbf{\Gamma}$ satisfy this property Φ. If such a result can be proved for all boldface pointclasses, it implies, in particular, that Borel sets satisfy property Φ.

The problem with this approach is that the games we need are not exactly the infinite games that fall under Definition 2.1.1. For example, the games may require the players to play other mathematical objects instead of natural numbers, and the winning condition may be rather complicated. To deal with this problem we must introduce the technique of *continuous coding* which allows us to treat complex rules as standard games falling under Definition 2.1.1. As motivation let us consider the following example (to which we will return in section 5.2. in more detail):

5.1.1 Definition. Let $A \subseteq \omega^\omega$ be a set, and consider the following game, called the *Banach-Mazur game* and denoted by $G^{**}(A)$: Players I and II alternate in taking turns, but instead of natural numbers, they play *non-empty sequences of natural numbers* $s_i, t_i \in \omega^{<\omega}$:

I:	s_0		s_1		$\dots$	
II:		t_0		t_1		$\dots$

Then let $z := s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \dots$ and say that Player I wins if and only if $z \in A$.

This game, at least on first sight, does not appear to be an infinite game according to our definition. However, as you may recall from our treatment of chess in the first chapter, the same was true there as well. What we did in order to formalize various games in a uniform way was to *code* the positions, moves etc. of the game as natural numbers. Clearly the same can be done here, too: since $\omega^{<\omega}$ is countable, we can fix a bijection $\varphi : \omega \longrightarrow \omega^{<\omega} - \{\langle\rangle\}$ and formulate the Banach-Mazur game as a game with natural numbers. But then, we must change the pay-off set accordingly: if A was the pay-off set of the Banach-Mazur game, let

$$A^{**} := \{z \in \omega^\omega : \varphi(z(0))^\frown\varphi(z(1))^\frown\varphi(z(2))^\frown \dots \in A\}$$

It is easy to see that Player I wins $G^{**}(A)$ if and only if he wins $G(A^{**})$, and the same holds for Player II. Hence the two games are equivalent, and we have succeeded in formalizing the Banach-Mazur game a game on natural numbers as in Definition 2.1.1.

However, one problem still remains. In applications, it is the Banach-Mazur game $G^{**}(A)$ we are interested in, not the encoded game $G(A^{**})$. Suppose we have a theorem proving that if $G^{**}(A)$ is determined, then A has some property Φ we are interested in. From that we wish to conclude that $\mathsf{Det}(\mathbf{\Gamma})$ implies that all sets in $\mathbf{\Gamma}$ satisfy property Φ. In other words, we would like to fix $A \in \mathbf{\Gamma}$ and conclude that $G^{**}(A)$ is determined. But we cannot! The only thing we can conclude is that $G(A)$ is determined, and that is not enough.

But we saw that $G^{**}(A)$ is equivalent to $G(A^{**})$, so *if* we knew that the set A^{**} is in $\mathbf{\Gamma}$, *then* we could conclude that $G(A^{**})$ is determined, hence that $G^{**}(A)$ is determined, and hence that A satisfies property Φ. But now, recall that our pointclass $\mathbf{\Gamma}$ was not just any collection of sets, but one satisfying two closure properties. If we could somehow use these closure properties to show that whenever $A \in \mathbf{\Gamma}$ then also $A^{**} \in \mathbf{\Gamma}$, we would be done. For example, if there is a closed set C and a continuous function f such that $A^{**} = f^{-1}[A] \cap C$ then it is clearly enough.

In general, we will usually have a situation similar to the one above, and we will need a combination of continuous functions and closed sets in order to reduce a complex game to a standard game. In a few cases we may need additional closure assumptions on the pointclass $\mathbf{\Gamma}$ but we will always make this explicit.

In each of the particular sections we will define special games, and verify that a sufficient coding indeed exists prior to using the game as a tool to prove some results.

5.2 The perfect set property

The questions in this section are motivated by an early attempt of Georg Cantor to solve the Continuum Hypothesis. We start with an important definition:

5.2.1 Definition. A tree T is called a *perfect tree* if every node $t \in T$ has at least two incompatible extensions in T, i.e., $\exists s_1, s_2 \in T$ such that $t \lhd s_1$ and $t \lhd s_2$ and $s_1 \perp s_2$.

Perfect trees also have an equivalent topological characterization: see Exercise 1.

5.2.2 Lemma. *If T is a perfect tree then $[T]$ has the cardinality of the continuum $2^{\aleph_0}$.*

Proof. We know that the set 2^ω of all infinite sequences of 0's and 1's has cardinality $2^{\aleph_0}$. Let $2^{<\omega}$ denote the collection of all finite sequences of 0's and

1's. Recall that a node $t \in T$ is called *splitting* if it has at least two successors $t^\frown \langle n \rangle$ and $t^\frown \langle m \rangle$ in T. By induction we define a function φ from $2^{<\omega}$ to the splitting nodes of T:

- $\varphi(\langle\rangle) :=$ least splitting node of T,
- If $\varphi(s)$ has been defined for $s \in 2^{<\omega}$, then $\varphi(s)$ is a splitting node of T, hence there are different n and m such that $\varphi(s)^\frown \langle n \rangle$ and $\varphi(s)^\frown \langle m \rangle$ are both in T. Now let $\varphi(s^\frown \langle 0 \rangle)$ be the least splitting node of T extending $\varphi(s)^\frown \langle n \rangle$ and $\varphi(s^\frown \langle 1 \rangle)$ be the least splitting node of T extending $\varphi(s)^\frown \langle m \rangle$.

We can now lift the function φ to $\hat{\varphi} : 2^\omega \longrightarrow [T]$ by setting

$$\hat{\varphi}(x) := \text{ the unique } z \in [T] \text{ such that } \forall s \lhd x \ (\varphi(s) \lhd z)$$

It only remains to verify that $\hat{\varphi}$ is injective. But if $x, y \in 2^\omega$ and $x \neq y$ then there is a least n such that $x \restriction n \neq y \restriction n$. But φ was inductively defined in such a way that $\varphi(x \restriction n) \neq \varphi(y \restriction n)$. Since $\varphi(x \restriction n) \lhd \hat{\varphi}(x)$ and $\varphi(t \restriction n) \lhd \hat{\varphi}(y)$, it follows that $\hat{\varphi}(x) \neq \hat{\varphi}(y)$.

Therefore there is an injection from 2^ω to $[T]$ and hence $[T]$ has cardinality $2^{\aleph_0}$. □

This theorem is much more intuitive then may seem at first sight. It simply says that if a set contains a "copy" of the full binary tree, then it must have the cardinality of the full binary tree, which is $2^{\aleph_0}$.

Now recall that the Continuum Hypothesis is the statement that every uncountable set has cardinality $2^{\aleph_0}$. Cantor intuitively wanted to prove this hypothesis along the following lines of reasoning: if a subset of the reals (or Baire space), is uncountable, then *there must be an explicit reason for it to be so.* The only explicit reason he could think of is that the set would contain a perfect tree. Hence, he defined the following dichotomy property for subsets of $\mathbb{R}$ (which we will present in the setting of ω^ω):

5.2.3 Definition. A set $A \subseteq \omega^\omega$ has the *perfect set property*, abbreviated by PSP, if it is either countable or there is a perfect tree T such that $[T] \subseteq A$.

Towards proving the Continuum Hypothesis, Cantor hoped to prove that *every* set satisfies the perfect set property. In fact, using the Axiom of Choice we can easily show that that assertion is false: the proof is analogous to the proof that there is a non-determined game, our Theorem 2.4.3. See Exercise 2. However, in this section we will show that it is reasonable for PSP to hold for sets in some limited pointclass, and the way we do that is by showing that PSP follows from determinacy. For that we will need to define the $*$-game, due to Morton Davis [Davis 1964].

5.2.4 Definition. Let $A \subseteq \omega^\omega$ be a set. The game $G^*(A)$ is played as follows:

- Player I plays non-empty sequences of natural numbers, and Player II plays natural numbers.

I :	s_0		s_1		s_2		$\dots$
II :		n_1		n_2		$\dots$	

- Player I wins $G^*(A)$ if and only if
    1. $\forall i \geq 1$: $s_i(0) \neq n_i$, and
    2. $x := s_0{}^\frown s_1{}^\frown s_2{}^\frown \dots \in A$.

In this game, the roles of I and II are not symmetrical. The intuition is that Player I plays finite sequences, attempting, in the limit, to form an infinite sequence in A. The outcome of the game depends only on the moves of Player I. The role of Player II is entirely different: the only influence she has on the game is that at each step she may choose a natural number n_i, and in the next move, Player I may play any sequence s_i whose first digit is not equal to n_i. Therefore I wins the game if he can overcome the challenges set by II and produce an infinite sequence in A. II wins if she can choose numbers in such a way as to prevent I from reaching his objective.

Before studying the consequences of the determinacy of $G^*(A)$, we must show that this game can be coded into a standard game, with moves in ω. Fix a bijection $\varphi : \omega \longrightarrow (\omega^{<\omega} - \langle\rangle)$. Then the $*$-game can be reformulated as a standard game with the pay-off set given by

$$A^* = \{z \in \omega^\omega : \forall n \geq 1\ (\varphi(z(2n))(0) \neq z(2n-1))$$
$$\wedge \quad \varphi(z(0))^\frown \varphi(z(2))^\frown \varphi(z(4))^\frown \dots \in A\}$$

It remains to show that for boldface pointclasses $\boldsymbol{\Gamma}$, $A \in \boldsymbol{\Gamma} \Rightarrow A^* \in \boldsymbol{\Gamma}$. For this we need two steps:

5.2.5 Lemma. *The function $f : \omega^\omega \longrightarrow \omega^\omega$ given by*

$$f(z) := \varphi(z(0))^\frown \varphi(z(2))^\frown \varphi(z(4))^\frown \dots$$

is continuous.

Proof. We must check that f satisfies the definition of continuity given in Definition 4.2.2. Fix z and let $s \lhd f(z)$. Let n be least such that $s \lhd \varphi(z(0))^\frown \dots {}^\frown \varphi(z(2n))$. Let $t := \langle z(0), z(1), \dots, z(2n)\rangle \lhd z$. Now, for any other y with $t \lhd y$ we have

$$f(y) = \varphi(z(0))^\frown \dots {}^\frown \varphi(z(2n))^\frown \varphi(y(2(n+1)))^\frown \dots$$

Therefore in any case $s \lhd f(y)$ holds, which completes the proof. ☐

5.2.6 Lemma. *The set $C := \{z \in \omega^\omega : \forall n \geq 1\ (\varphi(z(2n))(0) \neq z(2n-1))\}$ is closed.*

Proof. This is very similar to Exercise 4.10. Note that $C = \bigcap_{n \geq 1} C_n$ where

$$C_n := \{z : \varphi(z(2n))(0) \neq z(2n-1)\}$$

so it remains to show that each C_n is closed. But it is not hard to see that $C_n = [T_n]$ where $T_n := \{t \in \omega^{<\omega} : \text{if } |t| > 2n \text{ then } \varphi(t(2n))(0) \neq t(2n-1)\}$. We leave the details to the reader. ☐

Now it is clear that $A^* = C \cap f^{-1}[A]$, and since $\mathbf{\Gamma}$ is closed under continuous preimages and intersections with closed sets, we indeed see that $A \in \mathbf{\Gamma} \Rightarrow A^* \in \mathbf{\Gamma}$.

5.2.7 Theorem. (Davis, 1964) *Let $A \subseteq \omega^\omega$ be a set.*

1. *If Player I has a winning strategy in $G^*(A)$ then A contains a perfect tree.*
2. *If Player II has a winning strategy in $G^*(A)$ then A is countable.*

Proof.

1. Let σ be a winning strategy for Player I in the game $G^*(A)$. Although we are not talking about standard games, we can still use the notation $\sigma * y$ for an infinite run of the game in which II plays y and I according to σ, and similarly $\sigma * t$ for a finite position of the game. Let $\mathsf{Plays}^*(\sigma) := \{\sigma * y : y \in \omega^\omega\}$ as before and additionally let

   $$T_\sigma := \{s \in \omega^{<\omega} : s \lhd (\sigma * t) \text{ for some } t\}$$

   It is easy to verify that T_σ is a tree and that $[T_\sigma] = \mathsf{Plays}^*(\sigma) \subseteq A$. So it remains to show that T_σ is a perfect tree.

   Pick any $t \in T_\sigma$ and consider the least move i such that $t \lhd s_0 ^\frown \dots ^\frown s_i$. Now II can play n_{i+1} in her next move, after which I, assuming he follows σ, must play an s_{i+1} such that $s_{i+1}(0) \neq n_{i+1}$. Let $m_{i+1} := s_{i+1}(0)$. Instead of playing n_{i+1}, Player II could also have played m_{i+1} in which case I would have been forced to play a t_{i+1} such that $t_{i+1}(0) \neq m_{i+1}$. But then $t_{i+1} \neq s_{i+1}$, and both the sequence $s_0 ^\frown \dots ^\frown s_i ^\frown t_{i+1}$ and the sequence $s_0 ^\frown \dots ^\frown s_i ^\frown s_{i+1}$ are incompatible extensions of t according to σ, and hence are members of T_σ. This completes the proof that T_σ is perfect.

2. Now fix a winning strategy τ for II. Suppose p is a partial play according to τ, and such that it is Player I's turn to move, i.e., $p = \langle s_0, n_1, s_1, \dots, s_{i-1}, n_i \rangle$. Then we write $p^* := s_0 ^\frown \dots ^\frown s_{i-1}$. For such p and $x \in \omega^\omega$ we say:

   - p is *compatible with* x if there exists an s_i such that $s_i(0) \neq n_i$ and $p^* {}^\frown s_i \lhd x$. Note that this holds if and only if $p^* \lhd x$ and n_i (II's last move) doesn't "lie on x". Intuitively, this simply says that at position p, Player I still has a chance to produce x as the infinite play.

- *p rejects x* if it is compatible with x and maximally so, i.e., if for all s_i with $s_i(0) \neq n_i$, we have $p^\frown \langle s_i, \tau(p^\frown \langle s_i \rangle)\rangle$ is *not* compatible with x any longer. In other words, at position p Player I still has a chance to extend the game in the direction of x, but for just one more move, because, no matter which s_i he plays, Player II will reply with n_{i+1} according to her strategy τ, after which I will not have a chance to produce x any more.

Claim 1. *For every $x \in A$, there is a p which rejects it.*

Proof. Fix an $x \in A$ and towards contradiction, suppose there is no p which rejects it. Then at every stage of the game, Player I can play an s_i such that $s_0{}^\frown \dots {}^\frown s_i \lhd x$ and such that Player II's strategy τ can do nothing to stop him. That means there is a sequence y played by I such that $y * \tau = x \in A$, contradicting the fact that τ is a winning strategy for II. □

Claim 2. *Every p rejects at most one x.*

Proof. Suppose p rejects x and y and $x \neq y$. By definition, p is compatible with both x and y, so Player I can play some s_i with $s_i(0) \neq n_i$ and $p^{*\frown} s_i \lhd x$ and $p^{*\frown} s_i \lhd y$. But then, he can also play s_i to be *maximal* in this sense, i.e., such that any further extension $p^{*\frown} s_i{}^\frown \langle n \rangle$ cannot be an initial segment of both x and y (this is always possible since there is an n such that $x(n) \neq y(n)$).

Then consider $n_{i+1} := \tau(p^\frown \langle s_i \rangle)$. Clearly n_{i+1} cannot lie on both x and y, so $p^\frown \langle s_i, n_{i+1} \rangle$ can still be extended by Player I to be compatible with either x or y. Therefore, p does not reject both x and y □

If we now define $K_p := \{x \in \omega^\omega : p \text{ rejects } x\}$ we see that by Claim 1, $A \subseteq \bigcup_p K_p$, by Claim 2 each K_p is a singleton, and moreover there are only countably many p's. Hence A is contained in a countable union of singletons, so it is countable.

This completes the proof of the theorem. □

5.2.8 Corollary. $\mathsf{Det}(\mathbf{\Gamma})$ *implies that all sets in $\mathbf{\Gamma}$ have the perfect set property.*

Proof. Let A be a set in $\mathbf{\Gamma}$. Since A^* is also in $\mathbf{\Gamma}$, $G^*(A) = G(A^*)$ is determined. If Player I has a winning strategy in $G^*(A)$ then A contains a perfect tree, and if Player II has a winning strategy, then A is countable. □

5.3 The Baire property

In this section we study a property that has been important to topologists for a long time. The game used in this context is the Banach-Mazur game introduced in 5.1, and the method will be similar to the one from the previous section.

Recall the following topological definition: a set $X \subseteq \omega^\omega$ is *dense* if it intersects every non-empty open set. Of a similar nature are the following two definitions:

5.3.1 Definition. Let $X \subseteq \omega^\omega$. We say that

1. X is *nowhere dense* if every basic open $O(t)$ contains a basic open $O(s) \subseteq O(t)$ such that $O(s) \cap X = \varnothing$,
2. X is *meager* if it is the union of countably many nowhere dense sets.

An important aspect of the Baire space is that open sets cannot be meager[4] Meager sets are also called "of the first category" and, just as Lebesgue-null sets, can be considered "very small" or negligible in a topological sense. Because of this, it makes sense to talk about sets being equal "modulo meager". In particular, the following property is important:

5.3.2 Definition. A set $X \subseteq \omega^\omega$ has the *Baire property* if it is equal to an open set modulo meager, i.e., if there is an open set O such that $(X - O) \cup (O - X)$ is meager.

Just as with the perfect set property, it is possible to show (using the Axiom of Choice) that there are sets without the Baire property. We will prove that it follows from determinacy (for boldface pointclasses $\mathbf{\Gamma}$.)

The Banach-Mazur game $G^{**}(A)$ was already stated in Definition 5.1.1. The fact that we have continuous coding of the game depends on the following Lemma, which we leave as an exercise (it is very similar to Lemma 5.2.5).

5.3.3 Lemma. *Let φ be a bijection between ω and $\omega^{<\omega} - \{\langle\rangle\}$, and let f be the function from ω^ω to ω^ω defined by*

$$f(z) := \varphi(z(0))^\frown \varphi(z(1))^\frown \varphi(z(2))^\frown \dots$$

Then f is continuous.

Now it is clear that if $A \in \mathbf{\Gamma}$ then $A^{**} := f^{-1}[A] \in \mathbf{\Gamma}$, and $G^{**}(A)$ is equivalent to $G(A^{**})$.

[4]This is the so-called Baire category theorem. Sometime a topological space is called "a Baire space" if it satisfies this theorem, i.e., if open sets are not meager. In other spaces the property may fail.

5.3.4 Theorem.

1. *If Player II has a winning strategy in $G^{**}(A)$ then A is meager.*
2. *If Player I has a winning strategy in $G^{**}(A)$ then $O(s) - A$ is meager for some basic open $O(s)$.*

Proof.

1. Let τ be a winning strategy of Player II. For a position $p := \langle s_0, t_0, \dots, s_n, t_n \rangle$ write $p^* := s_0{}^\frown t_0{}^\frown \dots {}^\frown s_n{}^\frown t_n$. For any position p and $x \in \omega^\omega$ we say that

   - p is *compatible with* x if $p^* \lhd x$.
   - p *rejects* x if it is compatible and maximally so, i.e., if for any s_{n+1}, the next position according to τ, i.e., the position $p^\frown \langle s_{n+1}, \tau(p^\frown \langle s_{n+1} \rangle) \rangle$ is not compatible with x.

   Claim 1. *For every $x \in A$, there is a p which rejects x.*

   Proof. Just as in the proof of Theorem 5.2.7, if there were no p which rejected x then there is a sequence y of moves by Player I such that $x = y * \tau \in A$ contradicting the assumption that τ is winning for Player II. □

   Claim 2. *For every p, the set $F_p := \{x : p \text{ rejects } x\}$ is nowhere dense.*

   Proof. Let $O(s)$ be any basic open set. If $p^* \not\lhd s$ then we can extend s to some t such that any $x \in O(t)$ is incompatible with p^*, and hence not in F_p, i.e., $O(t) \cap F_p = \varnothing$. So the only interesting case is if $p^* \lhd s$. Now we use the following trick: suppose $p = \langle s_0, \dots, t_n \rangle$. Let s_{n+1} be the sequence such that $p^{*\frown} s_{n+1} = s$. Now let t_{n+1} be τ's answer, i.e., let $t_{n+1} := \tau(p^\frown \langle s_{n+1} \rangle)$. Then let $t := s^\frown t_{n+1}$. It is clear that $s \lhd t$ and hence $O(t) \subseteq O(s)$. We claim that $O(t) \cap F_p = \varnothing$ which is exactly what we need.

   Let $x \in O(t)$, i.e., $t \lhd x$. But if we look at the definition of rejection, it is clear that p cannot reject x, because for s_{n+1} Player II's response is t_{n+1} and the play $p^{*\frown} \langle s_{n+1}, t_{n+1} \rangle = t$ is compatible with x. Thus $x \notin F_p$. □

   Now the rest follows: by Claim 1, $A \subseteq \bigcup_p F_p$ which, by Claim 2, is a countable union of nowhere dense sets. Therefore A is meager.

2. Now we assume that Player I has a winning strategy σ in $G^{**}(A)$. Let s be I's first move according to the winning strategy, i.e., $s := \sigma(\langle\rangle)$. Then we claim:

   Claim 3. *Player II has a winning strategy in the game $G^{**}(O(s) - A)$*

   Proof. Here we shall see the first instance of how a player can translate an opponent's winning strategy into his own. We will describe the strategy informally.

   Let s_0 be I's first move in the game $G^{**}(O(s) - A)$.

   - **Case 1.** $s \not\lhd s_0$. Then play any t_0 such that $s_0{}^\frown t_0$ is incompatible with s. After that, play anything whatsoever. It is clear that the result of this game is some real $x \notin O(s)$, hence also not in $O(s) - A$, and therefore is a win for Player II.
   - **Case 2.** $s \lhd s_0$. Then let s_0' be such that $s^\frown s_0' = s_0$. Now Player II does the following trick: to determine her strategy she "plays another game on the side", a so-called *auxilliary game.* This auxilliary game is the original game $G^{**}(A)$ in which Player I plays according to his winning strategy σ. Player II will determine her moves based on the moves of Player I in the auxilliary game.

     The first move in the auxilliary game is $s := \sigma(\langle\rangle)$. Then Player II plays s_0' as the next move of the auxilliary game. To that, in the auxilliary game Player I responds by playing $t_0 := \sigma(\langle s, s_0'\rangle)$. Now Player II switches back to the "real" game, and copies that t_0 as her first response to I's "real" move, s_0.

     Next, in the "real" game she observes an s_1 being played by Player I. She then copies it as her next move in the auxilliary game, in which I responds according to σ with $t_1 := \sigma(\langle s, s_0', t_0, s_1\rangle)$. II copies t_1 on to the real game, and so it goes on. You can observe all this in the following diagram, where the first game represents the real game and the second the auxilliary one:

I:	s_0		s_1		$\dots$
II:		t_0		t_1	

I:	$s = \sigma(\langle\rangle)$		$t_0 = \sigma(\langle s, s_0'\rangle)$		$t_1 = \sigma(\langle s, s_0', t_0, s_1\rangle)$	
II:		s_0'		s_1		$\dots$

     In the final play of the real game, the infinite play

$$x := s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \dots$$

is produced. But clearly $x = s^\frown s'_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \dots$ and that was a play in the auxilliary game $G^{**}(A)$ in which Player I used his winning strategy σ. That means that $x \in A$. Therefore in the real game, $x \notin O(s) - A$ which means that the strategy which Player II followed was winning for her. And that completes the proof of Claim 3. □

Now it follows directly from part 1 of the theorem that $O(s) - A$ is meager, which is exactly what we had to show. □

After having proved the main theorem, we are close to the final result but not done yet. In fact what we have proven is that for boldface pointclasses $\mathbf{\Gamma}$, if $\mathsf{Det}(\mathbf{\Gamma})$ holds then for every $A \in \mathbf{\Gamma}$, either A is meager or $O(s) - A$ is meager for some basic open $O(s)$, which is a kind of "weak" Baire property. We will thus be done if we can prove the following final result.

5.3.5 Lemma. *Let $\mathbf{\Gamma}$ be a boldface pointclass. If for every $A \in \mathbf{\Gamma}$, either A is meager or $O(s) - A$ is meager for some $O(s)$, then every A in $\mathbf{\Gamma}$ satisfies the Baire property.*

Proof. Pick $A \in \mathbf{\Gamma}$. If A is meager we are done because A is equal to the open set $\varnothing$ modulo a meager set, hence has the Baire property. Otherwise, let

$$O := \bigcup\{O(s) : O(s) - A \text{ is meager }\}$$

This is an open set, and by definition $O - A$ is a countable union of meager sets, hence it is meager. It remains to show that $A - O$ is also meager. But since $\mathbf{\Gamma}$ is closed under intersections with closed sets, $A - O \in \mathbf{\Gamma}$. So if it is not meager, then there is $O(s)$ such that $O(s) - (A - O)$ is meager. That implies

1. $O(s) - A$ is meager, and
2. $O(s) \cap O$ is meager.

But the first statement implies, by definition, that $O(s) \subseteq O$. Then the second statement states that $O(s)$ is meager, and we know that that is false in the Baire space. So we conclude that $A - O$ is meager and so A has the Baire property. □

Combining everything in this section we get the following Corollary for boldface pointclasses $\mathbf{\Gamma}$.

5.3.6 Corollary. $\mathsf{Det}(\mathbf{\Gamma})$ *implies that every set in $\mathbf{\Gamma}$ has the Baire property.*

5.4 Flip sets

In this section we investigate another non-constructive object, the so-called "flip sets". For the time being we focus on the space 2^ω of all infinite sequences of 0's and 1's, rather than the Baire space. As 2^ω is a closed subspace of ω^ω, it is also a topological space with the subspace topology inherited from ω^ω, and the topology behaves exactly the same way (basic open sets are $O(t) := \{x \in 2^\omega : t \lhd x\}$).

5.4.1 Definition. An $X \subseteq 2^\omega$ is called a *flip set* if for all $x, y \in 2^\omega$, if x and y differ by exactly one digit, i.e., $\exists\,!\, n(x(n) \neq y(n))$, then

$$x \in X \iff y \notin X$$

Flip sets can be visualized by imagining an infinite sequence of light-switches such that flipping each switch turns the light on or off (in X or not in X). It is clear that if x and y differ by an even number of digits then $x \in X \iff y \in X$ whereas if they differ by an odd number then $x \in X \iff y \notin X$. If x and y differ by an infinite number of digits, we do not know what happens.

Although this gives us a very nice description of flip sets, it is not clear whether such sets exist. Indeed, if they do exist, then they are pathological in the sense of the space 2^ω. For example, it can be shown that flip sets are not Lebesgue measurable and do not have the Baire property. Flip sets can be constructed using the Axiom of Choice, and in this section we show that $\mathsf{Det}(\mathbf{\Gamma})$ implies that there are no flip sets in $\mathbf{\Gamma}$.

We consider a version of the Banach-Mazur game which is exactly as in Definition 5.1.1 but with I and II playing non-empty sequences of 0's and 1's. For a set $X \subseteq 2^\omega$, we denote the game by the same symbol $G^{**}(X)$. The fact that this can be coded using a continuous function is analogous to the previous case and we leave the details to the reader.

The way to prove that $\mathsf{Det}(\mathbf{\Gamma})$ implies that there are not flip sets in $\mathbf{\Gamma}$ is not by a direct application of determinacy, but rather by a sequence of Lemmas which, assuming a flip set exists in $\mathbf{\Gamma}$, lead to absurdity.

5.4.2 Lemma. *Let X be a flip set.*

1. *If I has a winning strategy in $G^{**}(X)$ then he also has a winning strategy in $G^{**}(2^\omega - X)$.*
2. *If II has a winning strategy in $G^{**}(X)$ then she also has a winning strategy in $G^{**}(2^\omega - X)$.*

Proof. Parts 1 and 2 are analogous so let us only do 1. If σ is a winning strategy for Player I in $G^{**}(X)$ then let σ' be as follows:

- The first move $\sigma'(\langle\rangle)$ is any sequence of the same length as $\sigma(\langle\rangle)$ and differs from it by exactly one digit.

- The next moves are played according to σ, pretending that the first move was $\sigma(\langle\rangle)$ and not $\sigma'(\langle\rangle)$.

It is clear that for any sequence y of II's moves, $\sigma * y$ and $\sigma' * y$ differ by exactly one digit. Since $\sigma * y \in X$ and X is a flip set, $\sigma' * y \notin X$, hence σ' is winning for I in $G^{**}(2^\omega - X)$. $\square$

5.4.3 Lemma. *Let X be a flip set. If II has a winning strategy in $G^{**}(X)$ then I has a winning strategy in $G^{**}(2^\omega - X)$.*

Proof. Let τ be II's winning strategy in $G^{**}(X)$. Player I does the following: first he play an arbitrary s. Player II will answer with some t. Now I starts playing an auxilliary version of $G^{**}(X)$ on the side, in which II uses τ. There he plays $s^\frown t$, and let s_0 be τ's answer in the auxilliary game. He copies s_0 as the next move in the real game. Player II will answer with some t_0. I copies t_0 on to the auxilliary game, etc.

$$G^{**}(2^\omega - X): \quad \begin{array}{r||ccccccc} \text{I:} & s & & s_0 & & s_1 & & \\ \hline \text{II:} & & t & & t_0 & & \dots & \end{array}$$

$$G^{**}(X): \quad \begin{array}{r||ccccccc} \text{I:} & s^\frown t & & t_0 & & t_1 & & \\ \hline \text{II:} & & s_0 & & s_1 & & \dots & \end{array}$$

Now if $x = s^\frown t^\frown s_0{}^\frown t_0{}^\frown \dots$ is the result of the game, it is the same as the result of the auxilliary game which was played according to τ. As τ was winning, it follows that $x \notin X$ and hence the strategy we just defined is winning for I in $G^{**}(2^\omega - X)$. $\square$

5.4.4 Lemma. *Let X be a flip set. If I has a winning strategy in $G^{**}(X)$ then II has a winning strategy in $G^{**}(2^\omega - X)$.*

Proof. This is slightly more involved then the previous two Lemmas. Let σ be winning for I in $G^{**}(X)$. Player II will, again, play two games: the main one $G^{**}(X)$, and an auxilliary $G^{**}(X)$ according to σ. Let Player I's first move in the real game be s_0. Let $s := \sigma(\langle\rangle)$ be I's first move in the auxilliary game. First, consider the case

- $|s_0| < |s|$.

Then in the real game, let II play t_0, such that $|s_0{}^\frown t_0| = |s|$ and $s_0{}^\frown t_0$ differs from s on an even number of digits. Clearly II can always find such t_0. Then let s_1 be I's next move in the real game. Player II copies it to the auxilliary game, in which I replies with some t_1, which II copies on to the real game, etc.

$$G^{**}(2^\omega - X): \quad \begin{array}{r||ccccccc} \text{I:} & s_0 & & s_1 & & s_2 & & \\ \hline \text{II:} & & t_0 & & t_1 & & \dots & \end{array}$$

$$G^{**}(X): \quad \begin{array}{r||ccccccc} \text{I:} & s & & & t_1 & & \dots & \\ \hline \text{II:} & & & s_1 & & s_2 & & \end{array}$$

Let $x := s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \dots$ be the result of the main game, and $y := s^\frown s_1{}^\frown t_1{}^\frown \dots$ the result of the auxilliary game. Then $y \in X$, and since by construction x and y differ by an even number of digits and X is a flip set, $x \in X$ follows, i.e., the strategy we described is winning for II in $G^{**}(2^\omega - X)$.

Now consider the case that

- $|s| \leq |s_0|$

This time Player II first plays any t in the auxilliary game such that $|s^\frown t| > |s_0|$, and finds s' to be Player I's reply in the auxilliary game. Now clearly $|s^\frown t^\frown t'| > |s_0|$ and she can play a t_0 in the real game such that $|s_0{}^\frown t_0| = |s^\frown t^\frown t'|$ and $s_0{}^\frown t_0$ and $s^\frown t^\frown t'$ differ on an even number of digits. After that she proceeds as before, i.e., if s_1 is the next move of I in the real game, she copies it into the auxilliary game, etc.

$G^{**}(2^\omega - X)$:

I:	s_0		s_1		s_2	
II:		t_0		t_1		$\dots$

$G^{**}(X)$:

I:	s		t'		t_1	$\dots$
II:		t		s_1		s_2

Now $x = s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \dots$ and $y = s^\frown t^\frown t'^\frown s_1{}^\frown t_1{}^\frown \dots$ differ by an even number of digits so the result follows. □

5.4.5 Theorem. *If* Det($\mathbf{\Gamma}$) *then there are no flip sets in* $\mathbf{\Gamma}$.

Proof. Suppose, towards contradiction, that there exists a flip set $X \in \mathbf{\Gamma}$. Then

- I has a winning strategy in $G^{**}(X) \implies$ I has a winning strategy in $G^{**}(2^\omega - X) \implies$ II has a winning strategy in $G^{**}(X)$, and
- II has a winning strategy in $G^{**}(X) \implies$ II has a winning strategy in $G^{**}(2^\omega - X) \implies$ I has a winning strategy in $G^{**}(X)$.

Both situations are clearly absurd, from which we conclude that there cannot be a flip set in $\mathbf{\Gamma}$. □

5.5 Wadge reducibility

Our last application of infinite games relates to Wadge reducibility, a large area of research of which we will only present a small part. The study of it started with the work of William Wadge (pronounced "wage") [Wadge 1983].

Since we will deal with complements of sets a lot in this section, it will be convenient to use the notation $\overline{A} := \omega^\omega - A$.

5.5.1 Definition. Let $A, B \subseteq \omega^\omega$. We say that A *is Wadge reducible to* B, notation $A \leq_W B$, if there is a continuous function f such that for all x:

$$x \in A \iff f(x) \in B$$

Clearly, $A \leq_W B$ if and only if $\overline{A} \leq_W \overline{B}$. Also, $\leq_W$ is reflexive and transitive, though in general not antisymmetric. Such relations are sometimes called *pre-orders*. We can make it into a partial order by taking equivalence classes: say A *is Wadge equivalent to* B, denoted by $A \equiv_W B$, if $A \leq_W B$ and $B \leq_W A$. Then let $[A]_W$ denote the equivalence class of A, i.e., $[A]_W := \{B \mid A \equiv_W B\}$. We can lift the ordering on to the equivalence classes: $[A]_W \leq_W [B]_W$ if and only if $A \leq_W B$, which is well-defined as can easily be verified. The equivalence classes $[A]_W$ are called *Wadge degrees* and the relation $\leq_W$ on the Wadge degrees is a partial order.

As usual, we define the strict Wadge ordering by setting

$$A <_W B \iff A \leq_W B \text{ and } B \not\leq_W A$$

Without determinacy, not much more can be said about the Wadge order. However, if we limit our attention on sets in a pointclass $\boldsymbol{\Gamma}$ satisfying $\mathsf{Det}(\boldsymbol{\Gamma})$, the picture changes entirely and a rich structure of the Wadge degrees emerges. Because of coding problems, in this section we will need additional requirements on $\boldsymbol{\Gamma}$, namely that it be closed under intersections and complements, i.e., if $A, B \in \boldsymbol{\Gamma}$ then $A \cap B \in \boldsymbol{\Gamma}$ and if $A \in \boldsymbol{\Gamma}$ then $\overline{A} \in \boldsymbol{\Gamma}$. The Borel sets, and all the $\boldsymbol{\Delta}^0_n$ and $\boldsymbol{\Delta}^1_n$ complexity classes satisfy this property, as well as the class of projective sets.

5.5.2 Definition. Let A, B be sets. The *Wadge game* $G^W(A, B)$ is played as follows: Players I and II choose natural numbers:

I:	x_0		x_1		$\dots$	
II:		y_0		y_1		$\dots$

If $x = \langle x_0, x_1, \dots \rangle$ and $y = \langle y_0, y_1, \dots \rangle$, then Player II wins $G^W(A, B)$ if and only if

$$x \in A \iff y \in B$$

To see that this game can be coded by a pay-off set in the same pointclass, consider the two functions f and g defined by

- $f(x)(n) := x(2n)$ and
- $g(x)(n) := x(2n + 1)$.

It is straightforward to verify that f and g are continuous. Now note that if z is the outcome of the Wadge game, then Player I wins $G^W(A, B)$ if and only if

$f(z) \in A \iff g(z) \notin B$. From this it follows that $G^W(A, B)$ is equivalent to the game

$$G((f^{-1}[A] - g^{-1}[B]) \cup (g^{-1}[A] - f^{-1}[B]))$$

and by our closure assumptions on $\mathbf{\Gamma}$, this set is also in $\mathbf{\Gamma}$, so our coding is adequate.

The main result is the following Theorem due to William Wadge.

5.5.3 Theorem. (Wadge, 1972) *Let* $A, B \subseteq \omega^\omega$.

1. *If Player II has a winning strategy in* $G^W(A, B)$ *then* $A \leq_W B$.
2. *If Player I has a winning strategy in* $G^W(A, B)$ *then* $B \leq_W \overline{A}$.

Proof.

1. Let τ be a winning strategy of II. For every x played by Player I, by definition of the winning condition, $x \in A \iff g(x * \tau) \in B$. But using the same methods as we have done many times before, it is easy to see that the function mapping x to $x * \tau$ is continuous. Similarly g is continuous, and so the composition of these two functions is a continuous reduction from A to B, so $A \leq_W B$.
2. Analogously, if σ is a winning strategy of I then for every y we have $f(\sigma * y) \in A \iff y \notin B$, and so again we have a continuous reduction from $\overline{B}$ to A, or equivalently from B to $\overline{A}$, so $B \leq_W \overline{A}$. □

Therefore, if we limit our attention to sets in $\mathbf{\Gamma}$ and assume $\mathsf{Det}(\mathbf{\Gamma})$, the Wadge order satisfies the property that for all A, B, either $A \leq_W B$ or $B \leq_W \overline{A}$. This immediately has many implications for the order. For example,

5.5.4 Lemma. *If* $A <_W B$ *then*

1. $A \leq_W B$,
2. $B \not\leq_W A$,
3. $A \leq_W \overline{B}$,
4. $B \not\leq_W \overline{A}$.

Proof. 1 and 2 are the definition, and 3 follows from 2 by the above Theorem. To see 4, suppose $B \leq_W \overline{A}$. Then by 1 we have $B \leq_W \overline{A} \leq_W \overline{B}$ and by the negation of 4 again, $B \leq_W \overline{A} \leq_W \overline{B} \leq_W A$. This contradicts 2. □

A set A, or its corresponding Wadge degree $[A]_W$, is called *self-dual* if $A \equiv_W \overline{A}$. Our characterization tells us the following:

5.5.5 Lemma. *If* A *is self-dual, then for any* B, *either* $B \leq_W A$ *or* $A \leq_W B$.

Proof. If $A \not\leq_W B$ then $B \leq \overline{A} \leq_W A$. □

We end this section, and with it the course, on the following involved proof of the Martin-Monk Theorem.

5.5.6 Theorem. (Martin-Monk) If $\mathsf{Det}(\mathbf{\Gamma})$ then the relation $<_W$ restricted to sets in $\mathbf{\Gamma}$ is well-founded.

Proof. We must show that there are no infinite descending $<_W$-chains of sets in $\mathbf{\Gamma}$. So, towards contradiction, suppose that $\{A_n : n \in \omega\}$ is an infinite collection of sets in $\mathbf{\Gamma}$ which forms an infinite descending $<_W$-chain:

$$\cdots <_W A_3 <_W A_2 <_W A_1 <_W A_0$$

Since for each n, $A_{n+1} <_W A_n$, by Lemma 5.5.4 (2) and (4), both $A_n \not\leq_W A_{n+1}$ and $A_n \not\leq_W \overline{A_{n+1}}$ hold. Therefore by Theorem 5.5.3 Player II cannot have a winning strategy in the games $G^W(A_n, A_{n+1})$ and $G^W(A_n, \overline{A_{n+1}})$. By determinacy, Player I must then have a winning strategy. We will call these strategies σ_n^0 and σ_n^1, respectively.

We now introduce the following abbreviation:

$$G_n^0 := G^W(A_n, A_{n+1})$$
$$G_n^1 := G^W(A_n, \overline{A_{n+1}})$$

Now to any infinite sequence of 0's and 1's, i.e., any $x \in 2^\omega$, we can associate an infinite sequence of Wadge games

$$\left\langle G_0^{x(0)}, G_1^{x(1)}, G_2^{x(2)}, \dots \right\rangle$$

played according to I's winning strategies

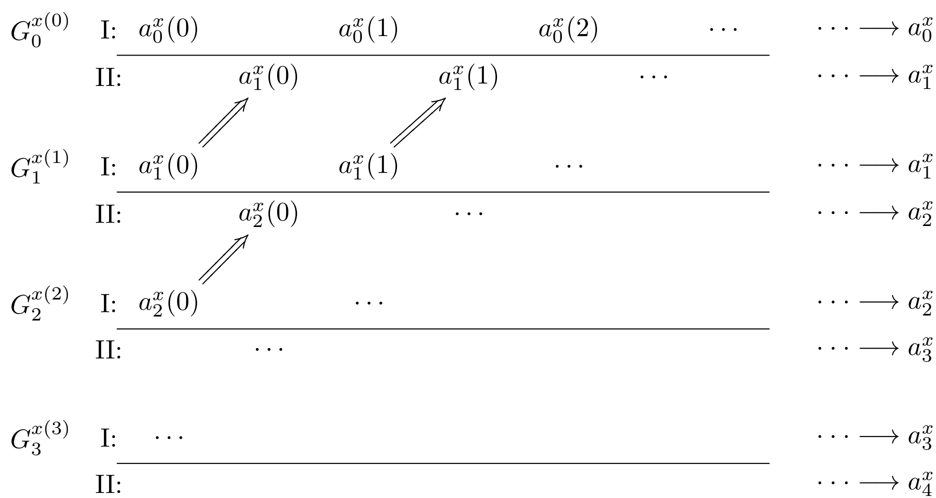
$$\left\langle \sigma_0^{x(0)}, \sigma_1^{x(1)}, \sigma_2^{x(2)}, \dots \right\rangle$$

Now we fix some particular $x \in 2^\omega$, and Player II is going to play all these infinite Wadge games simultaneously. In each game $G_n^{x(n)}$ Player I follows his winning strategy $\sigma_n^{x(n)}$, whereas Player II copies I's moves from the next game $G_{n+1}^{x(n+1)}$. To make this possible, she follows the following diagonal procedure:

- In the first game $G_0^{x(0)}$, let $a_0^x(0)$ be the first move of Player I, according to $\sigma_0^{x(0)}$. The superscript x refers to the infinite sequence we fixed at the start and the subscript 0 refers to the 0-th game.
- To play the next move in the first game, Player II needs information from the second game. Let $a_1^x(0)$ be Player I's first move in the game $G_1^{x(1)}$, according to $\sigma_1^{x(1)}$. Player II copies that move on to the first game.
- Next, Player I plays $a_0^x(1)$ in the first game. To reply to that, Player II needs information from the second game. There, $a_1^x(0)$ has been played, and Player II would like to copy information from the next game.

- So let $a_2^x(0)$ be Player I's first move in the game $G_2^{x(2)}$, according to $\sigma_2^{x(2)}$. Player II copies that on to the second game. Now $a_1^x(1)$ is I's next move in the second game, which Player II copies on to the first game.
- Etc.

All of this is best represented in the following diagram:



Using this procedure the two players are able to fill in the entire table. For each game $G_n^{x(n)}$ let a_n^x be the outcome of Player I's moves, and a_{n+1}^x be the outcome of Player II's moves. Note that the same infinite sequence a_{n+1}^x is also the result of I's moves in the next game, $G_{n+1}^{x(n+1)}$.

Since each game is won by Player I, the definition of the Wadge game implies that for each n:

$$\begin{array}{l} x(n) = 0 \implies (a_n^x \in A_n \ \leftrightarrow \ a_{n+1}^x \notin A_{n+1}) \\ x(n) = 1 \implies (a_n^x \in A_n \ \leftrightarrow \ a_{n+1}^x \in A_{n+1}) \end{array} \qquad (*)$$

Now we compare the procedure described above for different $x, y \in 2^\omega$.

Claim 1. *If* $\forall n \geq m \ (x(n) = y(n))$ *then* $\forall n \geq m \ (a_n^x = a_n^y)$.

Proof. Simply note that the values of a_n^x and a_n^y depend only on games $G_{n'}^{x(n')}$ and $G_{n'}^{y(n')}$ for $n' \geq n$. Therefore, if $x(n')$ and $y(n')$ are identical, so are the corresponding games and so are a_n^x and a_n^y. $\square$

Claim 2. *Let m be such that $\forall n \geq m\ (x(n) = y(n))$. Then $a^x_n \in A_n \leftrightarrow a^y_n \notin A_n$.*

Proof.

- **Case 1:** $x(n) = 0$ and $y(n) = 1$. By condition $(*)$ above it follows that

$$\begin{array}{l} a^x_n \in A_n \leftrightarrow a^x_{n+1} \notin A_{n+1} \\ a^y_n \in A_n \leftrightarrow a^y_{n+1} \in A_{n+1} \end{array}$$

  Since by Claim 1, $a^x_{n+1} = a^y_{n+1}$, it follows that

$$a^x_n \in A_n \leftrightarrow a^x_{n+1} \notin A_{n+1} \leftrightarrow a^y_{n+1} \notin A_{n+1} \leftrightarrow a^y_n \notin A_n$$

- **Case 2:** $x(n) = 1$ and $y(n) = 0$. Now $(*)$ implies that

$$\begin{array}{l} a^x_n \in A_n \leftrightarrow a^x_{n+1} \in A_{n+1} \\ a^y_n \in A_n \leftrightarrow a^y_{n+1} \notin A_{n+1} \end{array}$$

  Again by Claim 1 it follows that

$$a^x_n \in A_n \leftrightarrow a^x_{n+1} \in A_{n+1} \leftrightarrow a^y_{n+1} \in A_{n+1} \leftrightarrow a^y_n \notin A_n$$

□

Claim 3. *Let x and y be such that there is a unique n with $x(n) \neq y(n)$. Then $a^x_0 \in A_0 \leftrightarrow a^y_0 \notin A_0$.*

Proof. By Claim 2 we know that $a^x_n \in A_n \leftrightarrow a^y_n \notin A_n$. Since $x(n-1) = y(n-1)$ we again have two cases:

- **Case 1:** $x(n-1) = y(n-1) = 0$. Then by $(*)$

$$\begin{array}{l} a^x_{n-1} \in A_{n-1} \leftrightarrow a^x_n \notin A_n \\ a^y_{n-1} \in A_{n-1} \leftrightarrow a^y_n \notin A_n \end{array}$$

  and therefore $a^x_{n-1} \in A_{n-1} \leftrightarrow a^y_{n-1} \notin A_{n-1}$.

- **Case 2:** $x(n-1) = y(n-1) = 1$. Then by $(*)$ we have

$$\begin{array}{l} a^x_{n-1} \in A_{n-1} \leftrightarrow a^x_n \in A_n \\ a^y_{n-1} \in A_{n-1} \leftrightarrow a^y_n \in A_n \end{array}$$

  and therefore again $a^x_{n-1} \in A_{n-1} \leftrightarrow a^y_{n-1} \notin A_{n-1}$.

Now we go to the $(n-2)$-th level. Since again $x(n-2) = y(n-2)$ we get, by an application of $(*)$, that $a^x_{n-2} \in A_{n-2} \leftrightarrow a^y_{n-2} \notin A_{n-2}$. We go on like this until we reach the 0-th level, in which case we get $a^x_0 \in A_0 \leftrightarrow a^y_0 \notin A_0$, as required. □

Now define

$$X := \{x \in 2^\omega : a_0^x \in A_0\}$$

It is not hard to see that the map $x \longmapsto a_0^x$ is continuous: if we fix the first n values of a_0^x, we see that they only depend on the first n games $\{G_i^{x(i)} : i \leq n\}$. Therefore X is the continuous pre-image of A_0 and therefore $X \in \mathbf{\Gamma}$. But now Claim 3 says that X is a flip set, contradicting Theorem 5.4.5. $\square$

5.6 Exercises

1. Let X be a subset of a topological space. A point $x \in X$ is called an *isolated point of* X if there is an open set O such that $O \cap X = \{x\}$. Prove that a tree T is perfect if and only if $[T]$ has no isolated points.

2. Adapt the proof of Theorem 2.4.3 to prove, using AC, that there is a set that does not satisfy the Perfect Set Property.

3. (a) Show that for any set $A \notin \{\varnothing, \omega^\omega\}$, we have both $\varnothing <_W A$ and $\omega^\omega <_W A$.

   (b) Show that $\varnothing \not\leq_W \omega^\omega$ and $\omega^\omega \not\leq_W \varnothing$. Conclude that $[\varnothing]_W = \{\varnothing\}$ and $[\omega^\omega]_W = \{\omega^\omega\}$.

   (c) If $(P, \leq)$ is a partial order, then a subset $A \subseteq P$ is called an *antichain* if $\forall p, q \in A \ (p \not\leq q \wedge q \not\leq p)$. Show that in the partial order $(\mathbf{\Gamma}, \leq_W)$, assuming $\mathsf{Det}(\mathbf{\Gamma})$, antichains have size at most 2.

References

[Davis 1964] Morton Davis, *Infinite games of perfect information*, Advances in Game Theory, Annals of Mathematical Studies 52, Princeton University Press 1964, pp. 85–101.

[Gale & Stewart 1953] David Gale, F. M. Stewart, *Infinite games with perfect information*, Contributions to the Theory of Games II, ed. H. W. Kuhn and A. W. Tucker, Annals of Mathematics Studies 28, Princeton University Press (1953), pp 245–266.

[Schwalbe & Walker 2001] Ulrich Schwalbe, Paul Walker, *Zermelo and the Early History of Game Theory*, Games and Economic Behavior 34 (2001), pp. 123–137.

[Wadge 1983] William W. Wadge, *Reducibility and determinateness on the Baire space*, PhD thesis, University of California, Berkeley (1983).

[Zermelo 1913] Ernst Zermelo, *Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels*, Proceedings of the Fifth Congress of Mathematicians (Cambridge 1912), Cambridge University Press 1913, pp. 501-504